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Mathematical models of layered structures with an imperfect interface and delamination cracks

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The purpose of this thesis is to study layered structures with imperfect interfaces. There are several examples of structures where a main load-carrying component transfers its load to another through an adhesive thin layer such as aeroplane wing structures and automobile wind-screens. The problem includes the analysis of Boundary Value Problems (BVPs) of elastic thin structures bonded with a middle layer formally including two small parameters. This is because the adhesive joint is treated as a soft solid and infinitesimally thin in comparison with its adherends.

Propagation of waves on three-layered structures with an imperfect interface is studied. The asymptotic model introduced for the static problem is employed to yield a dispersion relation for the anti-plane shear problem.

As an illustration, the effect of having non-homogeneous conditions at the edges of the thin layered structure is studied. This enables one to study the boundary layer problem. The case for anti-plane shear is given.

Finally, mathematical models of cracks along imperfect interface boundaries have been analysed, in particular the asymptotic behaviour of the solution and its derivatives near the crack ends and at infinity. To be precise, analysis of the outer expansions of the solution is studied.This model arises from the study of the fracture of ceramic catalytic monolith combustors that are being incorporated into new proto-type designs of gas turbines. On the interface boundary (outside the crack) we allow for a non-zero displacement jump specified as a function of traction components. The presence of this condition affects the asymptotics of the displacement and stress components in the vicinity of the crack ends.

Keywords:

- adhesive joint,
- thin layered structure,
- displacement jump,
- delamination crack,
- propagation of waves.

To Olivia — for giving all her love and time.

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Chapter 1

Motivation, introduction and structure of the thesis

1.1 Motivation of the thesis

The motivation of this thesis is the enhancement of the performance of certain adhesive joints used in the design of aeroplane wing structures. The main objective of the thesis is to apply mathematical methods to the problems of layered structures with an imperfect interface and possibly delamination cracks. The problems involved are of three different kinds

- Static problems of orthotropic and isotropic single-lap joints,
- Dynamic problems of an isotropic single-lap joint,
- Delamination cracks on an imperfect interface.

The single-lap join problem (see Figure 1.1) has become increasingly important in recent years since advances in bonding techniques have made adhesive bonding in high loading situations possible. An adhesive is defined as a polymeric material which, when applied to surfaces such as metals, can join them together and resist separation. A structural adhesive, also referred to as an adhesive joint, is one used when the load required to cause separation is substantial such that the adhesive provides for the major strength and stiffness of the structure. The structural members of the joint, which are joined together by the adhesive, are called the adherends, a word first used by de Bruyne in 1939 (Adams and Wake, 1995).

The first adhesive joint successfully used was introduced in the aircraft industry for bonding aluminium to wood in 1944 (Adams and Wake, 1995). It was a

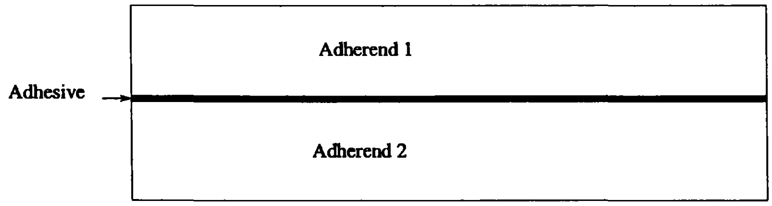

Figure 1.1: The single-lap joint.

polyvinyl formal composition with a phenol formaldehyde resin. Two years later in 1946, after the end of the second world war, the de Havilland Dove was the first all-metal passenger aircraft to be built (Bishop, 1997).

1.2 Background

The main subject of this investigation is layered structures and inhomogeneous media, a widely presented theme in mathematical and engineering literature. It is important to refer to some monographs where layered structures are studied Sendeckyj (1974); Christensen (1979); Jones (1975); Calcote (1969). Properties of laminates, as the optimal structures, have been considered by Milton (1986). Properties of different types of composite materials have been studied by Nemat-Nasser and Hori (1992). As for thin-walled composite media, the monograph by Kalamkarov and Kolpakov (1997) incorporates an engineering approach.

Most of the early work on anisotropic structures is based on the anisotropic equivalent (Lekhnitskii, 1963) of the traditional, but approximate, isotropic *thin-plate* theory (Love, 1927). This is termed the *classical laminate theory* when applied to the special case of laminated plates, and assumes the Kirchhoff-Love hypothesis of straight inextensional normals. It provides reasonably accurate predictions of the displacement field in thin plates. However it gives no information at all about the important interlaminar shear tractions. It does not include the study of layered structures with imperfect bonding.

The study of elastic thin layered structures with an imperfect interface is qualitatively different from the one for laminated structures with perfect bonding. We refer to Renardy and Rogers (1993) who give the mathematical theory of partial differential equations. The book by Ockendon *et al.* (1999) gives an impressive study of the applicability of the partial differential equations in the real world. General theory of elasticity can be found in the books by Muskhelishvili (1953) and Sokolnikoff (1956). The case when isotropic elastic plates with elastic properties varying arbitrarily through the thickness was studied by Spencer (1990). There is shown that a large class of exact solutions of the three-dimensional elasticity equations can be generated using the class of solution first introduced by Michell and the reader is referred to the monograph by Love (1927) where these solutions are described. For stretching deformations, the above solutions are generated by solutions of the classical thin plate equations for an equivalent plate whose elastic properties are appropriate averages of the elastic properties of the inhomogeneous plate.

The case for symmetric laminated plates comprising an arbitrary number of laminae of different isotropic materials each of uniform thickness with the laminae perfectly bonded at their interfaces was studied and some of the Michell's solutions generalised by Kaprielian *et al.* (1998). Their approach was to develop generalisations of Michell's solutions to apply in each individual lamina. These solutions were then shown to have sufficient generality to enable all the necessary interface continuity conditions to be satisfied, as well as the traction-free conditions on the faces. It was also shown that any solution of the equations of the two-dimensional classical laminate theory generates a corresponding three-dimensional solution for the laminate.

As for non-trivial boundary value problems in three-dimensional anisotropic elastic plate theory, papers by Rogers *et al.* (1992, 1995) and Spencer *et al.* (1993) present exact solutions for normal loading. They have used two mathematical methods: their formulation in terms of a transfer matrix and obtaining solutions in the form of asymptotic expansions.

Asymptotic methods for singularly perturbed problems have been extensively studied recently. The theory is based on the results of Kondrat'ev (1967) who analysed the behaviour of the solution of elliptic boundary value problems near conical and angular points. In Maz'ya *et al.* (1991), the method of compound asymptotic expansions in singularly perturbed boundary value problems is studied. Elliptic boundary value problems have been considered by Nazarov and Plamenevsky (1994) in domains with edges of different geometries.

For solutions of linear elasticity boundary value problems in thin domains, the monograph by Movchan and Movchan (1995) is referred. It also has an important basis to the asymptotic method for the study of layered structures with imperfect interfaces. Asymptotic analysis of linearly elastic shells is given by Miara and Sanchez-Palencia (1996). Stress-strain state of a plane with a thin elastic inclusion is given in Movchan and Nazarov (1988).

Joint design is dependent upon the nature of the materials to be joined as well as on the method of bonding. There is a close relationship between the materials to be joined, the nature of the stresses to be transmitted and the method of joining adopted. The examination of the relationship between joint strength and joint dimensions has been studied for many years. The early approach of Volkersen analysed the differential shear for a single-lap joint. That is, he considered the adhesive deforming only in shear with the adherends shearing only in tension (Adams and Wake, 1995). Modern digital computers have led to the use of a variety of numerical techniques for attempting to solve the adhesive joint problem. The finite-element technique has been used. However there is a problem since each solution applies only to a given set of parameters. It is necessary to vary joint length, adhesive thickness (which is very thin), adhesive modulus (which is much smaller that the one for the adherends) and so that a new processor is required each time. Thus, the computer power required is large and cost of a parametric study unattractive and not very accurate, particularly when the adhesive is considered to be soft and thin. The strength of adhesive joints using the theory of cracks was studied by Malyshev and Salganik (1965).

A first attempt to model this problem more accurately is given in Klarbring (1991), where an adhesive joint between two large solids is analysed by a variational based asymptotic method. Here, the adhesive is treated as a material surface. The only disadvantage at this point is that it is assumed a priori that the displacements vary linearly through the thickness of the adhesive and that certain terms of the resulting equations are negligible. A useful geometrical interpretation of the different deformation modes is given. Klarbring was the first who showed that in the adhesive layer problem there exists an explicit longitudinal displacement jump that dominates over other modes similar to the transverse displacement and the longitudinal displacement average.

Problems similar to the one treated here have been discussed by Caillerie (1980) and Suquet (1988). Nevertheless the asymptotic expansion method was not used. There the problem where the adhesive material is stiffer than that of the upper and lower layers is considered, contrary to the one to be discussed in this work. That is, they studied the problem where the Young's modulus E_0 characterising the adhesive behaves like $\epsilon^{-a}, a > 0$, as $\epsilon \to 0$.

The case when E_0 is independent of ϵ is briefly discussed by Nguetseng and Sanchez-Palencia (1985). Their result for the displacement field coincides with the one found by Caillerie (1980) for $0 < a < 1$, when the adhesive becomes a transmission in the limit, i.e. the two layers become one single layer.

As for boundary layer formulations of layered structures, it is worth mentioning papers by Baxter and Horgan (1995, 1997) who considered the end effects for anti-plane shear deformations of sandwich structures. More recently Benveniste

(1999) considered a similar formulation to Baxter and Horgan (1995) but he studied the end effects in conduction phenomena so that the middle thin layer was taken to be of low or high conductivity. Also we refer to the works by Zorin and Nazarov (1989) on the edge effect for the bending of a thin three-dimensional plate, Horgan (1995) on general theory of anti-plane shear deformations in linear and non-linear solid mechanics, Choi and Horgan (1977) on the Saint-Venant's principle and end effects in anisotropic elasticity and Crafter *et al.* (1993) who considered an semi-infinite, anisotropic elastic strip.

The case of plane deformations of sandwich strips with perfect bonding of isotropic phases was studied by Choi and Horgan (1978). We refer to the papers by Wijeyewickrema *et al.* (1996) on plane elasticity problems of sandwich structures; Horgan and Payne (1993) on the asymptotic behaviour of solutions of linear second-order boundary value problems on a semi-infinite strip; Horgan and Miller (1994) on anti-plane deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids and Horgan and Knowles (1983) on Saint-Venant's principle.

The modified interfacial conditions, that is, the condition of imperfect bonding for a sandwich structure has been included in papers by Benveniste (1984); Baxter and Horgan (1995) where the authors considered anti-plane deformations of anisotropic sandwich structures. Also Tullini *et al.* (1998) studied end effects for anti-plane shear deformations of periodically laminated strips with imperfect bonding. For the asymptotic analysis of fields in multi-structures, we refer to the recent monograph by Kozlov *et al.* (1999). They also studied boundary layer formulations for singularly perturbed problems.

The first results of this important consideration, when there exists an explicit longitudinal displacement jump derived without any physical assumptions, were obtained by Klarbring and Movchan (1995, 1998). Also the monograph by Movchan and Movchan (1995) provides the mathematical theory for analysing layered structures with imperfect interfaces. The problem includes the analysis of boundary value problems of elastic thin structures bonded with a middle layer formally including two small parameters. This is because the adhesive joint is treated as a soft solid and infinitesimally thin in comparison with its adherends. Different relationships between these parameters would lead to different governing equations for the layered structure. An asymptotic model of isotropic layered structures is fully developed and discussed. This technique enables one to characterise the layered structure with an imperfect interface by a small parameter. The novelty of this work is characterised by the fact that the limit problem of the asymptotic model allows for a displacement jump across the adhesive, and this jump condition is unknown. It is not specified a priori. This is a singularly perturbed problem due to the presence of the small parameter. The application of the finite element or finite difference methods requires a very fine mesh or in many cases this method fails. The convergence of such schemes is often unsatisfactory for the problems with small parameters. Asymptotic methods are free of these disadvantages.

It is virtually impossible to derive the correct model of an adhesive joint for an arbitrary relation between the parameters due to its softness and thinness on the basis of physical situation. This is a very important issue in the analysis of adhesive joints. Analysis of the boundary value problem describing an orthotropic laminated structure with imperfect interface has been given by Avila-Pozos *et al.* (1999).

The method of asymptotic expansions can be also used in homogenisation of composite materials where this technique allows one to derive the homogenised operator by considering multiple scale asymptotic expansions. In the monograph by Bensoussan *et al.* (1978), the multiple scale method is employed to treat boundary value problems with rapidly oscillating coefficients. Mathematical theory of homogenisation is given in the monograph by Jikov *et al.* (1995) where the method of asymptotic expansion has been used to justify homogenisation for second order elliptic operators with periodic coefficients. An averaging technique of defining the effective constants for layered and multi-rod structures has been considered by Bakhvalov and Panasenko (1989).

The propagation of mechanical disturbances in solids is of interest in many branches of the physical sciences and engineering. As elements of a medium are deformed, the disturbance is transmitted from one point to the next and the disturbance, or wave, progresses through the medium. There are many monographs on wave propagation in elastic solids and layered media. The reader is referred to the monographs by Achenbach (1973); Miklowitz (1980) and Brekhovskikh (1980).

Mathematical models for laminated media have been discussed extensively in the literature (see papers by Nayfeh (1975), Sun *et al.* (1968), Horvay *et al.* (1973) and Drumheller and Bedford (1973) for instance). The phenomenon of wave propagation in laminated and fibre reinforced composites was first studied by Hegemier *et al.* (1973).

The study of wave propagation in composites has attracted the interest of many people. However, the class of composite structures that has been treated is very restricted. Some of these structures consist of two-component structures combined in the form of either periodically laminated plates or fibre reinforced matrix materials. Wave propagation analysis in multi-component composites was first studied by Nayfeh (1974). A continuum mixture theory for horizontally polarised shear waves propagating along a three-layered structure has been given by Nayfeh and Loh (1977). There, the inner layer is treated as an agent bonding. Disregarding this crucial assumption, the softness of the material is not taken into account. Hence, it has been possible to derive a homogenised operator (see also papers by Nayfeh and Loh (1979) and Postma (1955)).

Dynamic problems of wave propagation in layered structures with a thin and soft interface layer are very important for industrial and military applications. Examples include delamination of components of body armour after an impact and optimal design of buildings resistant to earthquakes. Horizontally polarised waves in composites with bonds have been studied by Nayfeh and Nassar (1983). There it is assumed that the thickness of the bonding component is much smaller than the thicknesses of the outer layers. An additional assumption is made concerning the variation in displacement across the thickness of the bonding material. This variation or displacement jump is neglected.

The model proposed in this work is based on the asymptotic analysis applied to elasticity problems. We refer to the work by Avila-Pozos and Movchan (1999).

As for interfacial cracks, they have been studied by Erdogan (1963); England (1965); Erdogan (1965) and Rice and Sih (1965) who found the stresses around a crack on an interface between two isotropic elastic bodies under conditions of plane strain. After that, Willis (1971, 1972) solved the problem of a crack on an interface between two generally anisotropic materials and predicted the growth of a crack by applying Griffith's well-known virtual work argument. The effects near the vertex of a crack in a thin plate were studied by Nazarov (1991). The study of cracks in composite materials can be found in monographs by Sih (1978) and Sih and Chen (1981)

Monographs by Noble (1958) and Gakhov (1966) describe some methods to study mixed boundary value problems. All these methods are based on the complex variable theory. The work by Conway (1978) and Ahlfors (1966) give an exhaustive study of this subject.

The delamination crack along imperfect interfaces (represented by thin and soft layers) is a common phenomenon in many industrial applications. Examples include failure of laminated windscreens of modern cars and airplanes. The knowledge of the stress distribution near the delamination crack is essential for prediction of the critical load for a particular laminated structure. Until recently, the local analysis of gradient fields near cracks in adhesive joints (including the stress singularity) was an unanswered question. An efficient method based on analysis of the integral equation formulation and analytical solutions of appropriate problems of the Wiener-Hopf type has been developed in Antipov *et al.* (1999a). There an asymptotic analysis is applied to replace the interface layer with a set of effective contact conditions. The behaviour of the solution at the end of the crack is found by the same technique that was used by Antipov (1993).

1.3 Structure of the thesis

The structure of this thesis is as follows. The aim of Chapter 2 is to study layered structures including adhesives joints. There are several applications of this mathematical modelling such as windscreen in automobiles and the design of aeroplane wings. This problem is qualitatively different from the classical formulation for layered structures mentioned in the previous section and its solution is based on the compound asymptotic expansions constructed for thin regions (Nazarov, 1983b; Ciarlet, 1990; Movchan and Movchan, 1995; Arutyunyan *et al.*, 1987). The problem includes the analysis of boundary value problems (BVPs) of elastic thin structures bonded with a middle layer formally including two small parameters. This is because the adhesive joint is treated as a soft solid and infinitesimally thin in comparison with its adherends. The asymptotic method is used to derive the equations of the lower-dimensional structure. The asymptotic method enables one to reduce a complicated multi-parameter boundary value problem posed in a singularly perturbed domain to a set of model formulations independent of the small parameter and to construct an accurate asymptotic approximation for the solution. It also allows one to classify the multi-structures with imperfect interface layers and analyse the limit lower dimensional model problems. Anti-plane shear and plane elasticity problems are considered.

As for anti-plane shear, the resulting deformations are completely characterised by a single out-of-plane displacement that depends only on the in-plane coordinates. The non-homogeneous conditions at the edges of the thin layered structure enables one to study the boundary layer problem that will compensate for the error near the edges. Hence, the case for an anti-plane shear deformation is studied.

Chapter 3 deals with a three-layered structure in which the inner layer is modelled as an imperfect interface. We follow the pattern given by Klarbring and Movchan (1995); Movchan and Movchan (1995) and Klarbring and Movchan (1998). Here the materials are considered to be orthotropic. The limit lower dimensional equations describing the deformation of a thin layered orthotropic plate are derived as solvability conditions obtained for model problems on the cross-section. This method proves to be accurate and more efficient than approaches adopted in the existing engineering literature (Wooley and Carver, 1971). One advantage of the present asymptotic approach is that it enables one to reduce the original complicated three-dimensional problem of elasticity, posed in a region with a thin adhesive, to a set of differential equations on the limit surface. This derivation does not require any additional physical assumptions. Numerical experiments presented at the end of Chapter 3 demonstrate the potential for using the derived equations in large scale numerical computations. The limit equations here derived are valid within the three-dimensional region but near the edges they give an error that needs to be corrected (Zorin and Nazarov, 1989). A strength of materials approach is also considered in this to illustrate the asymptotic method.

Propagation of waves on three-layered structures with an imperfect interface is studied in Chapter 4. The analytical asymptotic model introduced in Chapters 2 and 3 is employed to yield a dispersion relation for the anti-plane shear problem (the following papers consider a similar geometry of the layered structure and simplify the effect of the bonding material: Nayfeh and Loh (1977); Nayfeh and Nassar (1978); Nayfeh (1980b)) and waves propagating along the imperfect interface. The analysis of this dispersion relation gives the information about the pass bands for the waves associated with the displacement jump across the interface.

The plane elasticity problem is also considered. This asymptotic method leads us to new limit equations, not previously studied, for stratified structures containing thin and soft adhesives. The range of applications include the effect of coupling between the fields associated with the presence of imperfect interfaces.

In Chapter 5 an integral equation method and the Wiener-Hopf technique are applied to the analysis of the new type of problems for cracks along an imperfect interface. Compared to classical formulations existing in the literature (Willis, 1971; England, 1965; Erdogan, 1965; Rice and Sih, 1965) at the end of the crack the conditions of ideal contact are not satisfied. The presence of the thin and soft layer implies that one can derive a so-called effective contact condition which, for the static case, would involve continuity of tractions and a certain relation between the traction components and the displacement jump. Motivation for this chapter arises from the study of the fracture of ceramic catalytic monolith combustors that are being incorporated into new proto-type designs of gas turbines. The ceramic monolith consists of an extruded structure that contains a large number of parallel channels, eg. consisting of: 62 cells / cm^2; each cell 1.1 mm $\times$1.1 mm square; with an open frontal area of 66%. The ceramic surface is coated with a high surface area material (eg. γ-Al_2O_3) which contains the dispersed catalyst. It is in the catalytic layer (also known as the wash-coat), where the combustion reactions take place (eg. $CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$), and the energy associated with this highly exothermic reaction is released. It is documented in engineering literature that the damage of ceramic structures is accompanied by *crack bridging*.

In the model presented here we assume that the bridging effect exists along the whole interface surface between the substrate and the layer of catalyst and, in addition, a crack with zero tractions on its faces is introduced along the interface contour. We study the problems of heat transfer (or anti-plane shear) and elasticity problems for this two-phase structure.

Analysis of cracks along imperfect interface boundaries and asymptotic behaviour of the solution and its derivatives near the crack ends and at infinity are considered. In contrast to the results already published in the literature, on the interface boundary and outside the crack we allow for a non-zero displacement jump specified as a function of traction components. The presence of this condition affects the asymptotics of the displacement and stress components in the vicinity of the crack ends.

1.4 Mathematical Methods I: Asymptotic approach

This section introduces the asymptotic algorithm that can be used to derive the equations of thin plates, shells and elastic beams. It will be used throughout this thesis.

1.4.1 Equations of linear elasticity in thin domains

Here, the solution of linear anisotropic elasticity in a thin region (for an isotropic thin beam the reader might want to see to Movchan and Movchan (1995)) is analysed. This derivation is based on the asymptotic analysis of the Lamé system in a thin domain and provides the justification of the engineering approach.

1.4.2 Thin elastic beam of a constant thickness

Here the equations of linear elasticity corresponding to the plane strain of a thin rectangular region Γ_ϵ are derived using an asymptotic method. For the leading order terms of the asymptotic approximation of the displacement field, the differential equations (that correspond to a bending or a longitudinal deformation) of a thin *anisotropic* beam are derived.

1.4.3 Plane strain state within a thin region

We consider a thin region Γ_ϵ of uniform thickness ϵh. Let the displacement vector $\boldsymbol{u}$ be defined as $\boldsymbol{u} = (u_1, u_2)^T$. The anisotropic response of the material is described by the stress-strain relations which can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_2 & c_4 & c_5 \\ c_3 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} \partial_1 u_1 \\ \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 \end{bmatrix} \tag{1.4.1}$$

where

$$\partial_i u_j \equiv \frac{\partial u_j}{\partial x_i}, \tag{1.4.2}$$

and c_1,c_2,... are the elastic moduli for the Ω_ϵ domain.

Assuming that the body forces are equal to zero, we write the equations of equilibrium in the form

$$\sum_{j=1}^{2} \partial_j \sigma_{ij} = 0. \tag{1.4.3}$$

Substituting the relation (1.4.1) into (1.4.3) we get

$$c_1 \partial_1^2 u_1 + (c_2 + c_6)\partial_{12}^2 u_2 + 2c_3 \partial_{12}^2 u_1 + c_3 \partial_1^2 u_2 + c_5 \partial_2^2 u_2 + c_6 \partial_2^2 u_1 = 0, \tag{1.4.4}$$
$$c_3 \partial_1^2 u_1 + 2c_5 \partial_{12}^2 u_2 + (c_2 + c_6)\partial_{12}^2 u_1 + c_6 \partial_1^2 u_2 + c_4 \partial_2^2 u_2 + c_5 \partial_2^2 u_1 = 0. \tag{1.4.5}$$

For the surfaces of the compound region Ω_ϵ we prescribe tractions:

$$\begin{aligned} c_3 \partial_1 u_1 + c_5 \partial_2 u_2 + c_6(\partial_2 u_1 + \partial_1 u_2) &= \epsilon^{-1} p_1^{(1)}, \\ c_2 \partial_1 u_1 + c_4 \partial_2 u_2 + c_5(\partial_2 u_1 + \partial_1 u_2) &= \epsilon^{-1} p_2^{(1)} \quad \text{on} \quad \Gamma_+ \end{aligned} \tag{1.4.6}$$

and

$$\begin{aligned} c_3 \partial_1 u_1 + c_5 \partial_2 u_2 + c_6(\partial_2 u_1 + \partial_1 u_2) &= \epsilon^{-1} p_1^{(2)}, \\ c_2 \partial_1 u_1 + c_4 \partial_2 u_2 + c_5(\partial_2 u_1 + \partial_1 u_2) &= \epsilon^{-1} p_2^{(2)} \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{1.4.7}$$

The edges of the layered structure are assumed to be fixed:

$$\boldsymbol{u}(\pm l, x_2) = \boldsymbol{0}.$$

If one introduces the variable

$$t = \frac{x_2}{\epsilon}, \tag{1.4.8}$$

one can see that

$$t \in [-h/2, h/2] \tag{1.4.9}$$

and

$$\partial_2 = \epsilon^{-1}\partial_t. \tag{1.4.10}$$

1.4.4 The formal asymptotic expansion

Let the displacement field be approximated by

$$\boldsymbol{u}(\boldsymbol{x}) \sim \sum_{k=0}^{\infty} \epsilon^k \left\{ \epsilon^{-4} \sum_{q=0}^{3} \epsilon^q \begin{pmatrix} v_1^{(q,k)}(x_1, t) \\ v_2^{(q,k)}(x_1, t) \end{pmatrix} \right.$$
$$\left. + \epsilon^{-2} \sum_{q=0}^{1} \epsilon^q \begin{pmatrix} \mathcal{V}_1^{(q,k)}(x_1, t) \\ \mathcal{V}_2^{(q,k)}(x_1, t) \end{pmatrix} + \begin{pmatrix} w_1^{(k)} \\ w_2^{(k)} \end{pmatrix} \right\}. \tag{1.4.11}$$

This asymptotic expansion was used by Movchan and Movchan (1995) for the isotropic case. This asymptotic representation is well-known and we refer to the works by Nazarov (1983b,a) where a more general problem is considered. There an asymptotic structure is proposed for elliptic operators in thin domains. Also papers by Arutyunyan *et al.* (1987) and Movchan and Nazarov (1988) consider elasticity problems for thin domains.

We will discuss the leading ansatz of the expansion (1.4.11) and derive the differential equations for the principal part of the displacement field. These differential equations will be the equations of an elastic *anisotropic* beam subjected to an external load applied on the upper and the lower surfaces. For convenience, we will omit the second superindex of the two first sums in curly brackets. This superindex is zero for the leading ansatz.

The first two sums in (1.4.11) provide the solvability of the problem with respect to $\boldsymbol{W}^{(0)} = (w_1^{(0)}, w_2^{(0)})^T$ obtained after substitution into the equations and the traction boundary conditions. These solvability conditions for $w_1^{(0)}$ and $w_2^{(0)}$ constitute a well-posed system for the principal part of the displacement field:

$$\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \begin{bmatrix} \partial_1^2 V_1^{(0)}(x_1) \\ \partial_1^4 v_2^{(0)}(x_1) \end{bmatrix} = \begin{bmatrix} G_1(x_1) \\ G_2(x_1) \end{bmatrix}, \tag{1.4.12}$$

where A_i, B_i are quantities depending on the thickness and the elastic moduli; $G_i(x_1)$ include the loading terms. Direct substitution of (1.4.11) into the system (1.4.4) and (1.4.5) with the traction boundary conditions (1.4.6) and (1.4.7) and equating the coefficients of powers of ϵ yields the following set of boundary value problems on $[-h/2, h/2]$:

Step 1 For $v_1^{(0)}$ and $v_2^{(0)}$ the following BVP is satisfied on the cross-section

$$\begin{aligned} c_6\partial_t^2 v_1^{(0)} + c_5\partial_t^2 v_2^{(0)} &= 0, \\ c_5\partial_t^2 v_1^{(0)} + c_4\partial_t^2 v_2^{(0)} &= 0 \quad \text{in } \Omega_\epsilon, \\ c_6\partial_t v_1^{(0)} + c_5\partial_t v_2^{(0)} &= 0 \quad \text{on } \Gamma_+, \Gamma_-, \\ c_5\partial_t v_1^{(0)} + c_4\partial_t v_2^{(0)} &= 0 \quad \text{on } \Gamma_+, \Gamma_-. \end{aligned} \tag{1.4.13}$$

The BVP (1.4.13) on the cross-section can be rewritten in terms of $v_1^{(0)}$ only as follows

$$\begin{aligned} D\partial_t^2 v_1^{(0)} &= 0, \text{ in } \Omega_\epsilon, \\ D\partial_t v_1^{(0)} &= 0, \text{ on } \Gamma_+, \Gamma_-, \end{aligned}$$

where $D = c_5c_5 - c_4c_6$. The solvability condition of the last BVP is identically satisfied.

Here it is considered that $v_1^{(0)} = 0$ as a result of the applied load (the transverse and longitudinal load have the same order of magnitude). In other words, the beam has much smaller resistance against the transverse load compared to the resistance against the longitudinal load.

Therefore, it can be seen that $v_2^{(0)}$ is a function of x_1 only.

Step 2 For $v_1^{(1)}$ and $v_2^{(1)}$ the following BVP on the cross-section can be established

$$
\begin{aligned}
c_6\partial_t^2 v_1^{(1)} + c_5\partial_t^2 v_2^{(1)} &= 0, & & \\
c_5\partial_t^2 v_1^{(1)} + c_4\partial_t^2 v_2^{(1)} &= 0 & &\text{in } \Omega_\epsilon, \\
c_6\partial_t v_1^{(1)} + c_5\partial_t v_2^{(1)} &= -c_6\partial_1 v_2^{(0)} & &\text{on } \Gamma_+, \Gamma_-, \\
c_5\partial_t v_1^{(1)} + c_4\partial_t v_2^{(1)} &= -c_5\partial_1 v_2^{(0)} & &\text{on } \Gamma_+, \Gamma_-.
\end{aligned}
\tag{1.4.14}
$$

The BVP (1.4.14) on the cross-section can be rewritten in terms of $v_2^{(1)}$ only as follows

$$
\begin{aligned}
D\partial_t^2 v_2^{(1)} &= 0, \text{ in } \Omega_\epsilon, \\
D\partial_t v_2^{(1)} &= 0, \text{ on } \Gamma_+, \Gamma_-.
\end{aligned}
$$

The solvability condition of the last BVP is identically satisfied. It follows that $v_2^{(1)}$ is a function of x_1 only. Since the structure of the ansatz in Relationship (1.4.11) repeats on every iteration of the asymptotic algorithm, the terms that depend on x_1 only are allocated for the beginning of every ansatz. Hence, it is assumed that the function $v_2^{(1)}$ is zero

$$v_2^{(1)} \equiv 0.$$

Thus, it can be seen that $v_1^{(1)}$ is given by

$$v_1^{(1)} = -t\partial_1 v_2^{(0)}.$$

Step 3 For $v_1^{(2)}$ and $v_2^{(2)}$ we have the following BVP on the cross-section

$$
\begin{aligned}
c_6\partial_t^2 v_1^{(2)} + c_5\partial_t^2 v_2^{(2)} &= c_3\partial_1^2 v_2^{(0)}, & & \\
c_5\partial_t^2 v_1^{(2)} + c_4\partial_t^2 v_2^{(2)} &= c_2\partial_1^2 v_2^{(0)} & &\text{in } \Omega_\epsilon, \\
c_6\partial_t v_1^{(2)} + c_5\partial_t v_2^{(2)} &= c_3 t\partial_1^2 v_2^{(0)} & &\text{on } \Gamma_+, \Gamma_-, \\
c_5\partial_t v_1^{(2)} + c_4\partial_t v_2^{(2)} &= c_2 t\partial_1^2 v_2^{(0)} & &\text{on } \Gamma_+, \Gamma_-.
\end{aligned}
\tag{1.4.15}
$$

Therefore one can see that

$$
\begin{aligned}
\partial_t^2 v_1^{(2)} &= \frac{c_2c_5 - c_3c_4}{D}\partial_1^2 v_2^{(0)} & &\text{in } \Omega_\epsilon, \\
\partial_t v_1^{(2)} &= \frac{(c_2c_5 - c_3c_4)t}{D}\partial_1^2 v_2^{(0)} & &\text{on } \Gamma_+, \Gamma_-.
\end{aligned}
\tag{1.4.16}
$$

The last BVP gives

$$v_1^{(2)} = \frac{(c_2c_5 - c_3c_4)t^2}{2D}\partial_1^2 v_2^{(0)}.$$

Similarly for $v_2^{(2)}$ we have the following BVP on the cross-section

$$\begin{aligned} \partial_t^2 v_2^{(2)} &= \frac{c_3c_5 - c_2c_6}{D}\partial_1^2 v_2^{(0)} \quad \text{in } \Omega_\epsilon, \\ \partial_t v_2^{(2)} &= \frac{(c_3c_5 - c_2c_6)t}{D}\partial_1^2 v_2^{(0)} \text{ on } \Gamma_+, \Gamma_- \end{aligned} \tag{1.4.17}$$

and

$$v_2^{(2)} = \frac{(c_3c_5 - c_2c_6)t^2}{2D}\partial_1^2 v_2^{(0)}.$$

Also it is seen that $\mathcal{V}_2^{(0)}$ can be assumed to be zero and $\mathcal{V}_1^{(0)}$ is a function of x_1 only. For convenience we write the following quantities:

$$\begin{aligned} E &= c_2c_5 - c_3c_4, \\ F &= c_3c_5 - c_2c_6. \end{aligned}$$

Step 4 For $U_1^{(3)}$ and $U_2^{(3)}$ we have the following BVP on the cross-section

$$\begin{aligned} c_6\partial_t^2 U_1^{(3)} + c_5\partial_t^2 U_2^{(3)} &= \frac{Dc_1 - (c_2 + c_6)F - 2c_3E}{D}t\partial_1^3 v_2^{(0)}, \\ c_5\partial_t^2 U_1^{(3)} + c_4\partial_t^2 U_2^{(3)} &= \frac{Dc_3 - (c_2 + c_6)E - 2c_5F}{D}t\partial_1^3 v_2^{(0)} \quad \text{in } \Omega_\epsilon, \\ c_6\partial_t U_1^{(3)} + c_5\partial_t U_2^{(3)} &= -\frac{(Ec_3 + Fc_6)t^2}{2D}\partial_1^3 v_2^{(0)} - c_3\partial_1 \mathcal{V}_1^{(0)}, \\ c_5\partial_t U_1^{(3)} + c_4\partial_t U_2^{(3)} &= -\frac{(Ec_2 + Fc_5)t^2}{2D}\partial_1^3 v_2^{(0)} - c_2\partial_1 \mathcal{V}_1^{(0)} \quad \text{on } \Gamma_+, \Gamma_- \end{aligned} \tag{1.4.18}$$

where $\boldsymbol{U}^{(3)} = \boldsymbol{v}^{(3)} + \boldsymbol{\mathcal{V}}^{(1)}$. Hence one can see that the following BVP on the cross-section is satisfied

$$\begin{aligned} \partial_t^2 U_1^{(3)} &= \left\{ \frac{c_5\{Dc_3 - (c_2 + c_6)E - 2c_5F\}}{D^2} \right. \\ &\quad \left. + \frac{-c_4\{Dc_1 - (c_2 + c_6)F - 2c_3E\}}{D^2} \right\} t\partial_1^3 v_2^{(0)} \quad \text{in } \Omega_\epsilon, \\ \partial_t U_1^{(3)} &= -\frac{(E^2 + DF)t^2}{2D^2}\partial_1^3 v_2^{(0)} - \frac{E}{D}\partial_1 \mathcal{V}_1^{(0)} \qquad \text{on } \Gamma_+, \Gamma_-. \end{aligned} \tag{1.4.19}$$

The last BVP gives

$$U_1^{(3)} = \{-E(E+I) - FG + DH - FD\}\frac{t^3}{6D^2}\partial_1^3 v_2^{(0)} + \left\{\frac{(EI+FG-DH)h^2}{8D^2}\partial_1^3 v_2^{(0)} - \frac{E}{D}\partial_1 \mathcal{V}_1^{(0)}\right\}t,$$

where for convenience we have written

$$G = c_5c_5 - c_2c_4,$$
$$H = c_3c_5 - c_1c_4,$$
$$I = c_5c_6 - c_3c_4.$$

Similarly for $U_2^{(3)}$ we have the following BVP on the cross-section

$$\begin{aligned}
\partial_t^2 U_2^{(3)} &= \left\{\frac{c_5\{Dc_1 - (c_2+c_6)F - 2c_3E\}}{D^2} - \frac{c_6\{Dc_3 - (c_2+c_6)E - 2c_5F\}}{D^2}\right\}t\partial_1^3 v_2^{(0)} \quad \text{in } \Omega_\epsilon, \\
\partial_t U_2^{(3)} &= -\frac{FEt^2}{2D^2}\partial_1^3 v_2^{(0)} - \frac{F}{D}\partial_1 \mathcal{V}_1^{(0)} \quad \text{on } \Gamma_+, \Gamma_-.
\end{aligned} \tag{1.4.20}$$

The last BVP gives

$$U_2^{(3)} = \{-E(F+M) + FL + DK\}\frac{t^3}{6D^2}\partial_1^3 v_2^{(0)} + \left\{\frac{(ME-DK-FL)h^2}{8D^2}\partial_1^3 v_2^{(0)} - \frac{F}{D}\partial_1 \mathcal{V}_1^{(0)}\right\}t, \tag{1.4.21}$$

where for convenience we have written

$$K = c_1c_5 - c_3c_6,$$
$$L = c_5c_6 - c_2c_5,$$
$$M = c_3c_5 - c_6c_6.$$

Step 5 Here we derive the ordinary differential equations with respect to $\mathbf{W}^{(0)} = (w_1^{(0)}, w_2^{(0)})^T$. Firstly for $w_1^{(0)}$ it is seen that the following BVP on the cross-section holds,

$$\begin{aligned}
D\partial_t^2 w_1^{(0)} &= -H\{\partial_1^2 v_1^{(2)} + \partial_1^2 V_1^{(0)}\} - I\partial_1^2 v_2^{(2)} \\
&\quad +\{(c_2+c_6)c_4 - 2c_5c_5\}\partial_{1t}^2 v_2^{(3)} \\
&\quad +\{2c_3c_4 - c_5(c_2+c_6)\}\partial_{1t}^2 v_1^{(3)}, && \text{in } \Omega_\epsilon, \\
D\partial_t w_1^{(0)} &= c_5 p_2^{(1)} - c_4 p_1^{(1)} - E\partial_1 v_1^{(3)} - D\partial_1 v_2^{(3)}, && \text{on } \Gamma_+, \\
D\partial_t w_1^{(0)} &= c_5 p_2^{(2)} - c_4 p_1^{(2)} - E\partial_1 v_1^{(3)} - D\partial_1 v_2^{(3)}, && \text{on } \Gamma_-.
\end{aligned} \tag{1.4.22}$$

Hence the last BVP has the following solvability condition

$$\boxed{\begin{aligned}
&A_1\partial_1^2 V_1^{(0)} + B_1\partial_1^4 v_2^{(0)} = c_5\{p_2^{(1)} - p_2^{(2)}\} + c_4\{p_1^{(2)} - p_1^{(1)}\}, \\
&A_1 = \{-HD + FG + EI\}\frac{h}{D}, \\
&B_1 = \left\{ G\{2(FL + DK - ME) - EF\} \right. \\
&\quad \left. +I\{2(DH - EI - FG) + EE\} - IFD \right\} \frac{h^3}{24D^2}.
\end{aligned}} \tag{1.4.23}$$

In a similar way we have that for $w_2^{(0)}$ the following BVP on the cross-section holds,

$$\begin{aligned}
D\partial_t^2 w_2^{(0)} &= -K\{\partial_1^2 v_1^{(2)} + \partial_1^2 V_1^{(0)}\} - M\partial_1^2 v_2^{(2)} \\
&\quad +\{2c_5c_6 - c_5(c_2+c_6)\}\partial_{1t}^2 v_2^{(3)} \\
&\quad +\{c_6(c_2+c_6) - 2c_3c_5\}\partial_{1t}^2 v_1^{(3)}, && \text{in } \Omega_\epsilon, \\
D\partial_t w_2^{(0)} &= c_5 p_1^{(1)} - c_6 p_2^{(1)} - F\partial_1 v_1^{(3)}, && \text{on } \Gamma_+, \\
D\partial_t w_2 &= c_5 p_1^{(2)} - c_6 p_2^{(2)} - F\partial_1 v_1^{(3)}, && \text{on } \Gamma_-.
\end{aligned} \tag{1.4.24}$$

Therefore the last BVP has the following solvability condition

$$\boxed{\begin{aligned}
&A_2\partial_1^2 V_1^{(0)} + B_2\partial_1^4 v_2^{(0)} = c_5(p_1^{(1)} - p_1^{(2)}) + c_6(p_2^{(2)} - p_2^{(1)}), \\
&A_2 = \{-KD + EM - LF\}\frac{h}{D}, \\
&B_2 = \left\{ L\{2(ME - FL - DK) - EF\} \right. \\
&\quad \left. +M\{2(DH - EI - FG) + EE\} - KED \right\} \frac{h^3}{24D^2}.
\end{aligned}} \tag{1.4.25}$$

The last two equations have been derived on the basis of the asymptotic analysis of the equilibrium equations in a thin rectangle without additional physical assumptions. For analysis of plane anisotropic beams we refer to the paper by Hashin (1967). There are of course monographs on the subject (see Lekhnitskii (1968) for example).

1.4.5 Orthotropic and isotropic cases

If the material is orthotropic then $c_3 = c_5 = 0$ and the limit equations (1.4.23)-(1.4.25) are given by

$$\frac{c_2^2 - c_1 c_4}{c_4} h \partial_1^2 v_1^{(0)} = p_1^{(1)} - p_1^{(2)}, \tag{1.4.26}$$

$$\frac{c_1 c_4 - c_2^2}{12 c_4} h^3 \partial_1^4 v_2^{(0)} = p_2^{(1)} - p_2^{(2)}. \tag{1.4.27}$$

These equations are well-known (see Lekhnitskii (1968)). One can note that the Equations (1.4.26) and (1.4.27) above coincide with the ones for the isotropic case if we impose $c_1 = c_4 = 2\mu + \lambda$, $c_2 = \lambda, c_6 = 2\mu$,

$$\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} h \partial_1^2 v_1^{(0)} = p_1^{(1)} - p_1^{(2)}, \tag{1.4.28}$$

$$\frac{\mu(\mu + \lambda)}{3(\lambda + 2\mu)} h^3 \partial_1^4 v_2^{(0)} = p_2^{(1)} - p_2^{(2)}. \tag{1.4.29}$$

1.4.6 A two-phase layered structure

In this section is presented the differential equations for a heterogeneous beam which includes two isotropic materials. The beam is assumed to be thin with thickness $\epsilon(h_1 + h_2)$ and the edges of the beam are clamped. It is seen that the beam can be referred to one that is homogeneous.

In this thesis we deal with the study of the behaviour of heterogeneous layered structures. To be precise, the study of the stiffness and strength of layered structures (the most important of all the engineering properties). The mathematical characterisation of the stiffness property is the basis of the methods of analysis that ultimately lead to design using engineering materials.

Here, we concentrate on determining the effective stiffness properties for a heterogeneous beam (it includes two layers with different isotropic properties). Effective stiffness properties refers to an average measure of the stiffness of the material, taking into account the properties of the two layers and their interaction.

In this case it is possible to obtain exact solutions for the effective moduli.

The condition of heterogeneity here is the change in properties across the interface between the two layers. The fundamental problem in this layered structure is to use a process to predict the effective properties of the idealised homogeneous medium in terms of the properties of each layer and the continuity condition at the interface. The relationship between the effective properties and the individual layer properties provides the means of optimising structural performance by varying the individual layer properties.

The variation of stress and strain through the layered structure thickness is essential to the definition of the extensional and bending stiffnesses of a heterogeneous beam. The two layers are perfectly bonded, i.e. the bond is presumed to be infinitesimally thin as well as non-shear-deformable. That is, the displacements are continuous across the interface, so both layers cannot slip relative to each other.

Let the displacement vector $\boldsymbol{u}$ be defined as $\boldsymbol{u} = (u_1, u_2)^T$. It can be seen (see Jones (1975) for instance) that the displacements satisfy the following differential equations:

$$4\left\{\frac{\mu_1(\lambda_1+\mu_1)}{2\mu_1+\lambda_1}h_1 + \frac{\mu_2(\lambda_2+\mu_2)}{2\mu_2+\lambda_2}h_2\right\}\partial_1^2 u_1(x_1) = p_1^{(1)} - p_1^{(2)}, \tag{1.4.30}$$

$$\frac{1}{3}\left\{\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^3 + \frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\right\}\partial_1^4 u_2(x_1) = p_2^{(1)} - p_2^{(2)}, \tag{1.4.31}$$

where $\mu_i, \lambda_i, i = 1, 2$ are the elastic moduli for the layers and we have taken into account the following tractions,

$$\begin{aligned} \mu_1(\partial_1 u_2 + \partial_2 u_1) &= \epsilon^{-1} p_1^{(1)}, \\ (2\mu_1+\lambda_1)\partial_2 u_2 + \lambda_1 \partial_1 u_1 &= \epsilon^{-1} p_2^{(1)} \quad \text{on} \quad \Gamma_+ \end{aligned} \tag{1.4.32}$$

and

$$\begin{aligned} \mu_2(\partial_1 u_2 + \partial_2 u_1) &= \epsilon^{-1} p_1^{(2)}, \\ (2\mu_2+\lambda_2)\partial_2 u_2 + \lambda_2 \partial_1 u_1 &= \epsilon^{-1} p_2^{(2)} \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{1.4.33}$$

The edges of the layered structure are assumed to be fixed:

$$\boldsymbol{u}(\pm l, x_2) = \mathbf{0}.$$

In Equation (1.4.30) the coefficient on the left-hand side is called extensional stiffness and in (1.4.31) the coefficient is called the bending stiffness. Now, it is seen that the equation (1.4.31) is similar to the one obtained for a single *isotropic* beam, i.e. it is the fourth order differential operator (see previous section). We can therefore consider the two-phased layered structure as a homogeneous layer of special elastic moduli.

1.5 Mathematical Methods II: Some complex variable methods

In this section, the theory used in Chapter 5 for a mathematical model of a delamination crack along a thin and soft interface layer is presented.

1.5.1 Some results of complex variable theory

Here the basic definitions of the complex variable theory are given. All these results are well-known and they can be found in Ahlfors (1966) and Conway (1978), for example.

If to each point ξ in an open set R of $\mathbb{C}$ there correspond one or more complex numbers, denoted by χ then we write $\chi = f(\xi)$ and say that χ is a function of the complex variable ξ. If the function has a uniquely defined value at each point of the region R it is said to be single-valued in R.

Definition. If R is an open set in $\mathbb{C}$ and $f : R \to \mathbb{C}$ then f is *differentiable* at a point ξ in R if

$$f'(\xi) = \lim_{\delta \to 0} \frac{f(\xi + \delta) - f(\xi)}{\delta}$$

exists and is independent of the direction along which the complex number δ tends to zero. The value of this limit is denoted by $f'(\xi)$ and is called the *derivative* of f at ξ. If f is differentiable at each point of R we say that f is *differentiable on* R.

Proposition. If a function $f : R \to \mathbb{C}$ is differentiable at a point ξ in R then f is continuous at ξ.

The function $\chi = f(\xi)$ is said to be *analytic* at the point ξ when it is single-valued and differentiable at this point. The function f is said to be *regular* in a region R if it is analytic at every point of R.

Points at which the function is not analytic are referred to as *singularities*. An analytic function which is regular in every finite region of the ξ-plane is called an *integral function*.

Theorem. *Liouville's theorem* is stated as follows. If $f(\xi)$ is an integral function such that $|f(\xi)| \leq M$ for all ξ, M being a constant then $f(\xi)$ is a constant.

1.5.2 Abelian theorems

In this section the Abelian theorems are stated. These theorems arise from asymptotic relations between some functions and their Fourier and Laplace transforms. The results of this section and the next one can be found in the monograph by Noble (1958). We first define

$$F_+(\alpha) = \int_0^\infty f(x)e^{i\alpha x}dx \quad ; \mathcal{F}(s) = \int_0^\infty f(x)e^{-sx}dx.$$

Then if, for $-1 < \eta < 0$,

$$f(x) \sim Ax^\eta, \quad x \to +0, \ x \to \infty,$$

we have

$$\begin{array}{lll} F_+(\alpha) \sim A\Gamma(\eta+1)e^{\frac{1}{2}\pi i(\eta+1)}\alpha^{-(\eta+1)}, & \alpha \to \infty, & \alpha \to +0, \\ \mathcal{F}(s) \sim A\Gamma(\eta+1)s^{-(\eta+1)}, & s \to \infty, & s \to 0, \end{array} \tag{1.5.1}$$

where upper and lower limiting processes go together. α tends to zero or infinity along paths in the upper half-plane, Im $\alpha > 0$; s tends to zero or infinity along the paths in the right half-plane, Re $s > 0$.

1.5.3 Plemelj formula

Here we consider the following integral

$$Q(\alpha) = \frac{1}{2\pi i}\int_C \frac{\phi(s)}{s-\alpha}ds, \tag{1.5.2}$$

for any complex α not on C. We assume that $\phi(s)$ satisfies a Hölder condition on C. That is,

$$|\phi(s_1) - \phi(s_2)| \leq M|s_1 - s_2|^\alpha,$$

for some $0 < \alpha \leq 1$ and M is a constant. Then $Q(\alpha)$ is regular everywhere in the α-plane except on C. We denote the limiting value of $Q(\alpha)$ as α tends to a point t of C from S^+ by $Q^+(t)$ and similarly for $Q^-(t)$. In addition we define direction on C so that S^+ lies on the left of C when C is transversed in the positive direction. In Relationship (1.5.2), we let α tend to a point t on C from

S^+ and from S^-. By deforming the contour suitably we find

$$Q^+(t) = \frac{1}{2}\phi(t) + \frac{1}{2\pi i}P\int_C \frac{\phi(s)}{s-t}ds, \tag{1.5.3}$$

$$Q^-(t) = -\frac{1}{2}\phi(t) + \frac{1}{2\pi i}P\int_C \frac{\phi(s)}{s-t}ds, \tag{1.5.4}$$

where the sign of P indicates that the integrals are now Cauchy principal values. By subtracting Relationship (1.5.4) from Relationship (1.5.3) we get the Plemelj formula

$$Q^+(t) - Q^-(t) = \phi(t). \tag{1.5.5}$$

1.5.4 Simple example: Mixed boundary value problem in a half-plane

In this section we solve the following elliptic mixed boundary value problem (see Ockendon *et al.* (1999))

$$\nabla^2 u(x,y) = 0, \;\; y > 0, \tag{1.5.6}$$

$$u(x,0) = 0, \quad x > 0, \tag{1.5.7}$$

$$\partial_y u|_{y=0} = 0, \quad x < 0. \tag{1.5.8}$$

We will assume that further in polar coordinates

$$u(r,\theta) = r^{-1/2}\sin(\theta/2) + O(r^{-3/2}), \;\; r^2 = x^2 + y^2 \to \infty. \tag{1.5.9}$$

and construct the crack face weight function.

1.5.5 Wiener-Hopf method

Although the solution of the problem (1.5.6)–(1.5.8) is trivial to recognise, to illustrate the Wiener-Hopf method we apply the Fourier transform with respect to x (this method is well-presented in the monograph by Noble (1958))

$$u_\alpha(y) = \int_{-\infty}^{+\infty} u(x,y)e^{i\alpha x}dx. \tag{1.5.10}$$

Here and in the text below the subscript index α denotes the Fourier transform. We obtain the ordinary differential equation

$$\frac{d^2u_\alpha}{dy^2} - \alpha^2 u_\alpha = 0, \tag{1.5.11}$$

which has the following solution

$$u_\alpha(y) = A(\alpha)e^{-|\alpha|y}.$$

We do not know A since we know neither u nor $\partial_y u$ for $y = 0$. We introduce the functions $f(x)$ and $g(x)$ such that

$$\partial_y u(x, 0) = f(x), \;\; u(x, 0) = g(x), \tag{1.5.12}$$

where

$$f(x) = \begin{cases} 0, & x < 0 \\ \text{unknown}, & x > 0 \end{cases}$$

and

$$g(x) = \begin{cases} \text{unknown}, & x < 0 \\ 0, & x > 0. \end{cases}$$

In other words, the closure of the set $\{x \in \mathbb{R} | f(x) \neq 0\} \subset [0, \infty)$. This set is named the support of $f(x)$ and is written supp f. Similarly supp $g \subset (-\infty, 0]$.

Therefore we obtain that

$$A(\alpha) = g_\alpha = -\frac{f_\alpha}{|\alpha|}. \tag{1.5.13}$$

Given that

$$f_\alpha = \int_0^\infty f(x)e^{i\alpha x}dx,$$

the integral exists and f_α is analytic if Im α is sufficiently large and positive, specifically in an upper half-plane. Also g_α is analytic if Im α is large enough and negative, specifically in a lower half-plane. Hence, there may be a line in the complex α-plane, probably Im α = constant, on which f_α and g_α are both analytic, or there may even be a strip where the regions of analyticity overlap. So, the Wiener-Hopf procedure is to write

$$f_\alpha = f_\alpha^+, \;\; g_\alpha = g_\alpha^-,$$

where the superscripts indicate the domain of analyticity. Then, we rewrite

$$f_\alpha^+ + |\alpha| g_\alpha^- = 0$$

in the following form

$$f_\alpha^+ + \frac{K_-}{K_+} g_\alpha^- = 0, \tag{1.5.14}$$

where the factors K_+ and K_- of $|\alpha|$ are not only analytic but also non-zero in upper and lower half-planes of the complex α-plane respectively. If the Relationship (1.5.14) holds, then the argument is as follows: $f_\alpha^+ K_+$ is analytic in an upper half-plane and its analytic continuation into the lower half-plane is equal to $-g_\alpha^- K_-$, which is known to be analytic there. Hence, both these equations must be entire. To find these entire functions, one uses Liouville's theorem as long as the behaviour of u at infinity is prescribed sufficiently carefully but the *factorisation* threatens the success of the whole procedure.

In order to write an arbitrary non-zero function $K(\alpha)$ as a ratio of "−" and "+" functions K_-/K_+, first we write

$$h_\alpha(\alpha) = -\log K(\alpha)$$

and consider

$$H(\alpha) = \frac{1}{2\pi i} \int \frac{h_\alpha(\alpha')}{\alpha' - \alpha} d\alpha'$$

along a contour Γ. Here α is a point in the strip and Γ is a small simple closed contour around α in the strip, executed in the counterclockwise sense. The contour is then *expanded to infinity* by means of Cauchy's theorem. So as to continue to satisfy the conditions of Cauchy's theorem, the contour leaves in its wake clockwise contours embracing all the poles and branch cuts of $H(\alpha)$ in the whole plane as the radius of the contour is increased to indefinitely large values. If the collection of such contours in Re $\alpha > \alpha_-$ is called Γ_- and the collection in Re $\alpha < \alpha_+$ is called Γ_+, then

$$H_\pm(\alpha) = \frac{1}{2\pi i} \int_{\Gamma_\pm} \frac{h_\alpha(\alpha')}{\alpha' - \alpha} d\alpha'.$$

The Cauchy integral defining $H_\pm$ is analytic everywhere except for α approaching certain points on $\Gamma_\pm$. In view of the fact that these paths are restricted to the appropriate half planes, the indicated regions of analyticity are obvious. Using the Plemelj formula, the difference between these two functions is $h_\alpha(\alpha)$. Hence we have identified two functions such that

$$h_\alpha(\alpha) = H_+ - H_-$$

and this is the *factorisation* for the function

$$K(\alpha) = \exp(-h_\alpha(\alpha)) = \frac{\exp H_+}{\exp H_-}.$$

In this particular case we can notice that

$$|\alpha| = \lim_{\epsilon\to 0} \sqrt{\alpha - i\epsilon}\sqrt{\alpha + i\epsilon}$$

where the branch cuts must lie along $(i\epsilon, i\infty)$ and $(-i\epsilon, -i\infty)$. Hence to within a multiplicative constant, we write

$$K_- = \sqrt{\alpha - i\epsilon}, \quad \arg\,(\infty - i\epsilon) \in (-\pi, 0),$$

where the branch cut is going from $i\epsilon$ to $+i\infty$ and

$$K_+ = \frac{1}{\sqrt{\alpha + i\epsilon}}, \quad \arg\,(\infty + i\epsilon) \in (0, \pi), \quad \mathrm{Im}\,\alpha \geq 0,$$

where the branch cut is going from $-i\epsilon$ to $-i\infty$, so that the function

$$f_\alpha^+ K_+ = -g_\alpha^- K_- = -g_\alpha^- \sqrt{\alpha - i\epsilon}$$

must be entire. Moreover the constraint on u as $r \to \infty$ gives that this entire function is bounded and hence constant. Thus, for some multiplicative constant $\mathcal{B}$,

$$g_\alpha^-(\alpha) = -\frac{\mathcal{B}}{\sqrt{\alpha - i\epsilon}}$$

Taking the inverse Fourier transform and using a suitable choice of $\mathcal{B}$ we get that

$$\boxed{g^- = r^{-1/2}, \quad x < 0.} \tag{1.5.15}$$

This singular solution is known as the crack face *weight function*.

Chapter 2

Asymptotic analysis of a sandwich beam in a state of anti-plane shear or plane strain

2.1 The adhesive joint structure

In this section the problem of a two-dimensional isotropic thin layered structure with an adhesive joint is formulated. This is qualitatively different from the case of a laminated structure with perfect bonding. This problem involves the analysis of boundary value problems of elastic thin structures bonded with a middle layer formally including two small parameters. This is due to the fact that the adhesive joint is treated as if it is a soft and infinitesimally thin solid in comparison with its adherends. Different relations between these parameters lead to different governing equations for the layered structure. A classification of limit equations obtained from varying these parameters is carried out. The works by Klarbring (1991); Horgan and Miller (1994); Hashin (1967), Basi *et al.* (1991), Pagano (1970); Nguetseng and Sanchez-Palencia (1985) and Suquet (1988) are relevant to this chapter since they deal with a similar layered structure with an imperfect interface.

Recently, Klarbring and Movchan (1998) studied a boundary value problem for the Laplacian that corresponds to the case of an anti-plane shear of the layered structure with an adhesive (thin and soft interface layer). They performed an asymptotic analysis of a two-parameter perturbation problem. For the analysis of the asymptotic expansions for homogeneous structures, including the general ansatz, we refer to the work of Nazarov (1983b,a).

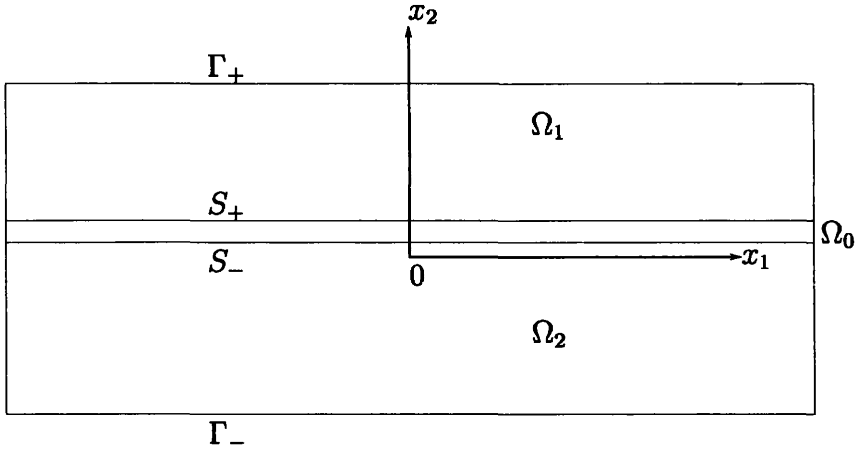

Figure 2.1: The two-dimensional thin region Ω_ϵ.

In this Chapter is emphasised that the relationship between the two small parameters is important and gives the key to providing several kinds of limit operators.

Consider a thin rectangular domain, which consists of three layers as shown in Figure (2.1),

$$\begin{aligned}
\Omega_1 &= \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < l\,,\ \epsilon(h/2 - h_1) + \epsilon^2 h_0 < x_2 < \epsilon h/2 + \epsilon^2 h_0\}, \\
\Omega_2 &= \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < l\,,\ -\epsilon h/2 < x_2 < -\epsilon h/2 + \epsilon h_2\}, \\
\Omega_0 &= \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < l\,,\ -\epsilon(h/2 - h_2) < x_2 < -\epsilon(h/2 - h_2) + \epsilon^2 h_0\},
\end{aligned}$$

where l and h_i , $i = 0, 1, 2$, have the same order of magnitude, and ϵ is a small non-dimensional positive parameter; h is defined by $h = h_1 + h_2$.

The upper and lower layers have the thickness ϵh_1 and ϵh_2, respectively, and the middle layer is thinner (thickness $\epsilon^2 h_0$). The interface boundary includes two parts, S_+ and S_-, specified by

$$\begin{aligned}
S_+ &= \{\boldsymbol{x} : |x_1| < l\,,\ x_2 = -\epsilon(h/2 - h_2) + \epsilon^2 h_0\}, \\
S_- &= \{\boldsymbol{x} : |x_1| < l\,,\ x_2 = -\epsilon(h/2 - h_2)\}.
\end{aligned} \tag{2.1.1}$$

The upper and lower surfaces of the compound region are

$$\begin{aligned}
\Gamma_+ &= \{\boldsymbol{x} : |x_1| < l\,, x_2 = \epsilon^2 h_0 + \epsilon h/2\}, \\
\Gamma_- &= \{\boldsymbol{x} : |x_1| < l\,, x_2 = -\epsilon h/2\}.
\end{aligned}$$

The layers Ω_1 and Ω_2 are adhesively connected by Ω_0. Introducing the non-dimensional variables

$$\begin{aligned} t_0 &= \epsilon^{-2}(x_2 + \epsilon(h/2 - h_2) - \epsilon^2 h_0/2), \\ t_1 &= \epsilon^{-1}(x_2 - \epsilon^2 h_0 - \epsilon h_2/2), \\ t_2 &= \epsilon^{-1}(x_2 + \epsilon h_1/2), \end{aligned} \tag{2.1.2}$$

one can see that

$$t_j \in [-h_i/2, h_i/2], \;\; i = 1, 2; \;\; t_0 \in [-h_0/2, h_0/2] \tag{2.1.3}$$

and

$$\partial_2 = \epsilon^{-2}\partial_{t_0}, \;\; \partial_2 = \epsilon^{-1}\partial_{t_i}, \;\; i = 1, 2, \tag{2.1.4}$$

where the notation ∂_α is the same as used in the previous chapter.

2.2 Anti-plane shear

First, we study a BVP for the Laplacian that corresponds to the case of an anti-plane shear of the layered structure. We consider different values of the normalised shear modulus of the middle layer. To simplify the final formulae it is assumed that the quantities h_0, h_1 and h_2 are the same for all three layers

$$H := h_0 = h_1 = h_2. \tag{2.2.1}$$

The displacement vector depends on x_1 and x_2 only and has the form

$$\boldsymbol{u}^{(i)} = (0, 0, w^{(i)}(x_1, x_2)),$$

where the functions $w^{(i)}, i = 0, 1, 2$ satisfy the Poisson equations

$$\mu_i \Delta w^{(i)}(x_1, x_2) + f_i(x_1, x_2) = 0, \;\; \text{in } \Omega_i, \;\; i = 0, 1, 2 \tag{2.2.2}$$

and the homogeneous Neumann boundary conditions on the sides $\Gamma_\pm$

$$\frac{\partial w^{(i)}}{\partial x_2} = 0 \;\; \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \tag{2.2.3}$$

where $\mu_i, i = 0, 1, 2$ are the shear moduli of the materials. It is assumed here that $f_0 \equiv 0$. We shall use the superscript index notation $w^{(i)}$ to denote the displacement in the region Ω_i. On the interfaces $S_\pm$, displacement continuity conditions are prescribed

$$w^{(1)} = w^{(0)},$$
$$\mu_1 \frac{\partial w^{(1)}}{\partial x_2} = \mu_0 \frac{\partial w^{(0)}}{\partial x_2} \quad \text{on } S_+, \tag{2.2.4}$$
$$w^{(0)} = w^{(2)},$$
$$\mu_2 \frac{\partial w^{(1)}}{\partial x_2} = \mu_0 \frac{\partial w^{(0)}}{\partial x_2} \quad \text{on } S_-. \tag{2.2.5}$$

The ends $x_1 = \pm l$ of the compound beam are assumed to be fixed such that

$$w^{(i)}(x_1 = \pm l, x_2) = 0. \tag{2.2.6}$$

We seek the asymptotic approximation for the functions $w^{(i)}, i = 0, 1, 2$ in the form

$$w^{(i)}(\boldsymbol{x}) \sim W^{(i,0)}(x_1, t_i) + \epsilon W^{(i,1)}(x_1, t_i) + \epsilon^2 W^{(i,2)}(x_1, t_i), \; i = 0, 1, 2. \tag{2.2.7}$$

The middle layer is normalised in such a way that

$$\mu_0 = \epsilon^m \mu, \tag{2.2.8}$$

where m is positive integer and the quantity μ has the same order of magnitude as μ_1 and μ_2. Putting the last series (2.2.7) into system (2.2.2) and equating the coefficients of like powers of the small parameter ϵ, the following recurrent system of BVPs on the cross-section of the compound beam can be obtained,

$$\frac{\partial^2 W^{(i,k)}}{\partial t_i^2} = -\frac{\partial^2 W^{(i,k-2)}}{\partial x_1^2} - \frac{f_i \delta_{k2}}{\mu_i} \quad \text{in } \Omega_i, i = 1, 2 \tag{2.2.9}$$

and

$$\frac{\partial^2 W^{(0,k)}}{\partial t_0^2} = -\frac{\partial^2 W^{(0,k-4)}}{\partial x_1^2} \quad \text{in } \Omega_0. \tag{2.2.10}$$

Also the boundary conditions (2.2.3) become

$$\frac{\partial}{\partial t_1} W^{(1,k)} = 0 \quad \text{on } \Gamma_+, \tag{2.2.11}$$
$$\frac{\partial}{\partial t_2} W^{(2,k)} = 0 \quad \text{on } \Gamma_-. \tag{2.2.12}$$

The interface boundary conditions, given by relations (2.2.4) and (2.2.5), can be written as

$$W^{(1,k)} = W^{(0,k)}, \ \mu_1 \partial_{t_1} W^{(1,k)} = \mu \partial_{t_0} W^{(0,k-m+1)} \ \text{ on } S_+, \tag{2.2.13}$$
$$W^{(2,k)} = W^{(0,k)}, \ \mu_2 \partial_{t_2} W^{(2,k)} = \mu \partial_{t_0} W^{(0,k-m+1)} \ \text{ on } S_-, \tag{2.2.14}$$

where the index m is the same as in (2.2.8), $k = 0, 1, 2$, and all terms with negative indices vanish. Thus, we have some recurrent sequences of the Neumann BVPs on the cross-section for ordinary differential equations in the upper and lower layers (the longitudinal variable x_1 is included as a parameter) and the sequence of Dirichlet BVPs for the middle layer. The Neumann BVPs mentioned require certain solvability conditions for the right-hand sides of the equations and boundary conditions. As a result, the limit equation for the leading term of the approximation (2.2.7) can be found. Here, a classification of problems for different values of m in the relationship (2.2.8) is given. For convenience the following notation shall be used

$$W^{+(1,k)} = W^{(1,k)}(x_1, \frac{-h_1}{2}), \ \ W^{-(2,k)} = W^{(2,k)}(x_1, \frac{h_2}{2}). \tag{2.2.15}$$

2.2.1 Limit equation when $\mu_0 = \epsilon\mu$

Step 1 If $m = 1$, in relations (2.2.9), (2.2.11)–(2.2.14) it is seen that for $k = 0$ the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned}
\partial^2_{t_i} W^{(i,0)} &= 0, && \text{in } \Omega_i, \ i = 1, 2, \\
\partial_{t_i} W^{(i,0)} &= 0, && \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\
\partial_{t_i} W^{(i,0)} &= \frac{\mu}{\mu_i} \partial_{t_0} W^{(0,0)}, && \text{on } S_+(i = 1), S_-(i = 2).
\end{aligned} \tag{2.2.16}$$

It follows immediately that the solvability conditions for the BVP (2.2.16) are given by

$$\partial_{t_0} W^{+(0,0)} = \partial_{t_0} W^{-(0,0)} = 0. \tag{2.2.17}$$

Also, with the boundary conditions, the first terms of the approximation (2.2.7), for both the upper and lower layers, are found to be functions of x_1 only,

$$W^{(i,0)} \equiv W^{(i,0)}(x_1), i = 1, 2.$$

For the middle layer, Equations (2.2.10), (2.2.13) and (2.2.14) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 W^{(0,0)} &= 0, && \text{in } \Omega_0, \\ W^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= W^{(1,0)}, && \text{on } S_+, \\ W^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= W^{(2,0)}, && \text{on } S_-. \end{aligned}$$

Hence, the representation of $W^{(0,0)}(x_1, t_0)$ can be written in the form

$$W^{(0,0)}(x_1, t_0) = A^{(0)}(x_1) + \frac{t_0}{h_0} D^{(0)}(x_1), \tag{2.2.18}$$

where $A^{(0)}(x_1)$ represents the average of the functions $W^{(1,0)}(x_1)$ and $W^{(2,0)}(x_1)$ evaluated at the interfaces S_+ and S_-,

$$A^{(0)}(x_1) = \frac{W^{+(1,0)}(x_1) + W^{-(2,0)}(x_1)}{2},$$

whereas $D^{(0)}(x_1)$ is referred to as the displacement jump evaluated at the interfaces S_+ and S_-,

$$D^{(0)}(x_1) = W^{+(1,0)}(x_1) - W^{-(2,0)}(x_1).$$

Using the relationship (2.2.18), the solvability condition (2.2.17) can be written in the form

$$W^{(1,0)}(x_1) = W^{(2,0)}(x_1) = W^{(0,0)}(x_1) \equiv W^{(0)}(x_1).$$

Step 2 Making the substitution $k = 1$ in relation (2.2.9), with boundary conditions (2.2.11)–(2.2.14) it is seen that the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 W^{(i,1)} &= 0, && \text{in } \Omega_i,\ i = 1, 2, \\ \partial_{t_i} W^{(i,1)} &= 0, && \text{on } \Gamma_+(i=1), \Gamma_-(i=2), \\ \partial_{t_i} W^{(i,1)} &= \frac{\mu}{\mu_i} \partial_{t_0} W^{(0,1)}, && \text{on } S_+(i=1), S_-(i=2). \end{aligned} \tag{2.2.19}$$

For both BVPs (2.2.19) $i = 1$ and $i = 2$ the solvability conditions give

$$\partial_{t_0} W^{+(0,1)} = \partial_{t_0} W^{-(0,1)} = 0. \tag{2.2.20}$$

For the middle layer, Equations (2.2.10), (2.2.13) and (2.2.14) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 W^{(0,1)} &= 0, && \text{in } \Omega_0, \\ W^{(0,1)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= W^{(1,1)}, && \text{on } S_+, \\ W^{(0,1)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= W^{(2,1)}, && \text{on } S_-. \end{aligned}$$

Therefore, the representation of $W^{(0,1)}(x_1, t_0)$ can be written in the form

$$W^{(0,1)}(x_1, t_0) = A^{(1)}(x_1) + \frac{t_0}{h_0} D^{(1)}(x_1), \tag{2.2.21}$$

where $A^{(1)}(x_1)$ represents the average of the functions $W^{(1,1)}(x_1)$ and $W^{(2,1)}(x_1)$ evaluated at the interfaces S_+ and S_-,

$$A^{(1)}(x_1) = \frac{W^{+(1,1)}(x_1) + W^{-(2,1)}(x_1)}{2},$$

while $D^{(1)}(x_1)$ is referred to as the displacement jump evaluated at the interfaces S_+ and S_-,

$$D^{(1)}(x_1) = W^{+(1,1)}(x_1) - W^{-(2,1)}(x_1).$$

Using the relation (2.2.21) for $W^{(0,1)}(x_1, t_0)$ it is concluded that the solvability condition (2.2.20) can be written as

$$W^{(1,1)}(x_1) = W^{(2,1)}(x_1) = W^{(0,1)}(x_1) \equiv W^{(1)}(x_1).$$

Step 3 The notation $t_1 = t_2 \equiv t$ will be used as a result of the simplification established by the relation (2.2.1). Substituting $k = 2$ in relation (2.2.9), for $i = 1$ with boundary conditions (2.2.11) and (2.2.13) it is seen that the following BVP on the cross-section holds for the upper layer,

$$\begin{aligned} \partial_{t_1}^2 W^{(1,2)} &= -\partial_1^2 W^{(0)} - \frac{f_1}{\mu_1}, && \text{in } \Omega_1, \\ \partial_{t_1} W^{(1,2)} &= 0, && \text{on } \Gamma_+, \\ \partial_{t_1} W^{(1,2)} &= \frac{\mu}{\mu_1} \partial_{t_0} W^{(0,2)}, && \text{on } S_+. \end{aligned} \tag{2.2.22}$$

For the BVP (2.2.22) the following solvability condition must be satisfied

$$\partial_1^2 W^{(0)}(x_1) = \frac{\mu}{\mu_1 H^2}\partial_{t_0} W^{+(0,2)} - \frac{1}{\mu_1 H}\int_{-H/2}^{H/2} f_1 d_{t_1}. \tag{2.2.23}$$

For $i = 2$ with $k = 2$ substituted in relation (2.2.9) and boundary conditions (2.2.12),(2.2.14) it is seen that the following BVP on the cross-section holds for the lower layer,

$$\begin{aligned} \partial_{t_2}^2 W^{(2,2)} &= -\partial_1^2 W^{(0)} - \frac{f_2}{\mu_2}, && \text{in } \Omega_2, \\ \partial_{t_2} W^{(2,2)} &= 0, && \text{on } \Gamma_-, \\ \partial_{t_2} W^{(2,2)} &= \frac{\mu}{\mu_2}\partial_{t_0} W^{(0,2)}, && \text{on } S_-,. \end{aligned} \tag{2.2.24}$$

For the BVP (2.2.24) the following solvability condition must be satisfied,

$$\partial_1^2 W^{(0)}(x_1) = -\frac{\mu}{\mu_2 H^2}\partial_{t_0} W^{-(0,2)} - \frac{1}{\mu_2 H}\int_{-H/2}^{H/2} f_2 d_{t_2}. \tag{2.2.25}$$

For the middle layer, Equations (2.2.10), (2.2.13) and (2.2.14) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 W^{(0,2)} &= 0, && \text{in } \Omega_0, \\ W^{(0,2)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= W^{(1,2)}, && \text{on } S_+, \\ W^{(0,2)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= W^{(2,1)}, && \text{on } S_-. \end{aligned}$$

Hence, the representation of $W^{(0,2)}(x_1, t_0)$ can be written in the form

$$W^{(0,2)}(x_1, t_0) = A^{(2)}(x_1) + \frac{t_0}{h_0} D^{(2)}(x_1), \tag{2.2.26}$$

where $A^{(2)}(x_1)$ represents the average of the functions $W^{(1,2)}(x_1)$ and $W^{(2,2)}(x_1)$ evaluated at the interfaces S_+ and S_-,

$$A^{(2)}(x_1) = \frac{W^{+(1,2)}(x_1) + W^{-(2,2)}(x_1)}{2},$$

while $D^{(2)}(x_1)$ is referred to as the displacement jump evaluated at the interfaces S_+ and S_-,

$$D^{(2)}(x_1) = W^{+(1,2)}(x_1) - W^{-(2,2)}(x_1).$$

Using the relationship (2.2.26), the solvability conditions (2.2.23) and (2.2.25) are rewritten as

$$\partial_1^2 W^{(0)}(x_1) = \frac{\mu}{\mu_1 H^2} D^{(2)}(x_1) - \frac{1}{\mu_1 H} \int_{-H/2}^{H/2} f_1 d_t, \tag{2.2.27}$$

$$\partial_1^2 W^{(0)}(x_1) = -\frac{\mu}{\mu_2 H^2} D^{(2)}(x_1) - \frac{1}{\mu_2 H} \int_{-H/2}^{H/2} f_2 d_t. \tag{2.2.28}$$

Using the Equations (2.2.27) and (2.2.28), the displacement jump $D^{(2)}(x_1)$ can be represented as

$$D^{(2)}(x_1) = -\frac{\mu_1 \mu_2 H}{\mu(\mu_1 + \mu_2)} \int_{-H/2}^{H/2} \left(\mu_2^{-1} f_2(x_1, t) - \mu_1^{-1} f_1(x_1, t)\right) dt \tag{2.2.29}$$

and

$$\boxed{\partial_1^2 W^{(0)}(x_1) = -\frac{1}{(\mu_1 + \mu_2) H} \int_{-H/2}^{H/2} \left(f_1(x_1, t) + f_2(x_1, t)\right) dt.} \tag{2.2.30}$$

The discrepancy of order $O(1)$ in the boundary conditions at $x_1 = \pm l$ will be removed if

$$\boxed{W^{(0)}(\pm l) = 0.} \tag{2.2.31}$$

Thus, the displacement jump across the thin layer is insignificant in this case, while the leading order displacement $W^{(0)}(x_1)$ is the same in all three layers and it satisfies the one-dimensional boundary value problem (2.2.30)–(2.2.31). Equation (2.2.30) is the limit equation for the thin layered structure under the deformation of an anti-plane shear for the case when $\mu_0 = \epsilon\mu$. The limit equation (2.2.30) is similar to the one already available in literature. The reader is referred to the monographs by Movchan and Movchan (1995), Lekhnitskii (1968) and the papers by Horgan and Miller (1994) and Horgan and Payne (1993).

2.2.2 Limit equation when $\mu_0 = \epsilon^2\mu$

For this case, the analysis is almost exactly the same as for $\mu_0 = \epsilon\mu$. The only difference is that the displacement jump will have the order $O(\epsilon)$ instead of $O(\epsilon^2)$. With this, the solution $W^{(0)}$ for all three layers is the same and thus satisfies the one-dimensional boundary value problem (2.2.30)–(2.2.31).

2.2.3 Limit equation when $\mu_0 = \epsilon^3 \mu$

Step 1 If $m = 3$ in relations (2.2.9), (2.2.11)–(2.2.14), it is seen that for $k = 0$ the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial^2_{t_i} W^{(i,0)} &= 0, \quad \text{in } \Omega_i,\ i = 1, 2, \\ \partial_{t_i} W^{(i,0)} &= 0, \quad \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} W^{(i,0)} &= 0, \quad \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{2.2.32}$$

The solvability conditions for the BVPs (2.2.32) are identically satisfied. Moreover, with the boundary conditions, the first terms of the approximation (2.2.7), for both the upper and lower layers, are found to be functions of x_1 only,

$$W^{(i,0)} \equiv W^{(i,0)}(x_1), i = 1, 2.$$

For the middle layer, Equations (2.2.10), (2.2.13) and (2.2.14) give the Dirichlet BVP (2.2.18) on the cross-section. Therefore, the representation of $W^{(0,0)}(x_1, t_0)$ is given by Equation (2.2.18).

Step 2 Making the substitution $k = 1$ in relation (2.2.9), with boundary conditions (2.2.11)–(2.2.14) it is seen that the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial^2_{t_i} W^{(i,1)} &= 0, \quad \text{in } \Omega_i,\ i = 1, 2, \\ \partial_{t_i} W^{(i,1)} &= 0, \quad \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} W^{(i,1)} &= 0, \quad \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{2.2.33}$$

For both BVP (2.2.33) the solvability conditions are identically satisfied as in the previous step. The representations of $W^{(i,1)}, i = 1, 2$ are given in terms of x_1 only. For the middle layer, Equations (2.2.10), (2.2.13) and (2.2.14) give the Dirichlet BVP (2.2.21) on the cross-section. Hence, the representation of $W^{(0,1)}(x_1, t_0)$ can be expressed by Relationship (2.2.21).

Step 3 Substituting $k = 2$ in relation (2.2.9), for $i = 1$ with boundary conditions (2.2.11), (2.2.13) it is seen that the following BVPs on the cross-section for the upper layer can be established,

$$\begin{aligned} \partial^2_{t_1} W^{(1,2)} &= -\partial^2_1 W^{(1,0)} - \frac{f_1}{\mu_1}, && \text{in } \Omega_1, \\ \partial_{t_1} W^{(1,2)} &= 0, && \text{on } \Gamma_+, \\ \partial_{t_1} W^{(1,2)} &= \frac{\mu}{\mu_1}\partial_{t_0} W^{(0,0)}, && \text{on } S_+. \end{aligned} \tag{2.2.34}$$

For the BVP (2.2.34) the following solvability condition must be satisfied

$$\partial^2_1 W^{(1,0)}(x_1) = \frac{\mu}{\mu_1 H^2}\partial_{t_0} W^{+(0,0)} - \frac{1}{\mu_1 H}\int_{-H/2}^{H/2} f_1(x_1, t) d_t. \tag{2.2.35}$$

For $i = 2$ with $k = 2$ substituted in relation (2.2.9) and boundary conditions (2.2.12),(2.2.14) it is seen that the following BVP on the cross-section holds for the lower layer,

$$\begin{aligned} \partial^2_{t_2} W^{(2,2)} &= -\partial^2_1 W^{(2,0)} - \frac{f_2}{\mu_2}, && \text{in } \Omega_2, \\ \partial_{t_2} W^{(2,2)} &= 0, && \text{on } \Gamma_-, \\ \partial_{t_2} W^{(2,2)} &= \frac{\mu}{\mu_2}\partial_{t_0} W^{(0,0)}, && \text{on } S_-. \end{aligned} \tag{2.2.36}$$

For the BVP (2.2.36) the following solvability condition must be satisfied,

$$\partial^2_1 W^{(2,0)}(x_1) = -\frac{\mu}{\mu_2 H^2}\partial_{t_0} W^{-(0,0)} - \frac{1}{\mu_2 H}\int_{-H/2}^{H/2} f_2 d_t. \tag{2.2.37}$$

For the middle layer, Equations (2.2.10), (2.2.13) and (2.2.14) give the Dirichlet BVP (2.2.26) on the cross-section. Hence, the representation of $W^{(0,2)}(x_1, t_0)$ can be expressed in the form (2.2.26). If one substitutes the representation of $W^{(0,0)}(x_1, t_0)$ given in the relationship (2.2.18) into Equations (2.2.35), (2.2.37) then one can write

$$\boxed{\begin{aligned} \partial^2_1 W^{(1,0)}(x_1) &= \frac{\mu}{\mu_1 H^2} D^{(0)}(x_1) - \frac{1}{\mu_1 H}\int_{-H/2}^{H/2} f_1(x_1, t) d_t, \\ \partial^2_1 W^{(2,0)}(x_1) &= -\frac{\mu}{\mu_2 H^2} D^{(0)}(x_1) - \frac{1}{\mu_2 H}\int_{-H/2}^{H/2} f_2(x_1, t) d_t. \end{aligned}} \tag{2.2.38}$$

Equations (2.2.38) are the limit equations of the thin layered structure including only terms of the leading order. Here the term $D^{(0)}(x_1)$ denotes the displacement jump across the thin layer and has the form

$$D^{(0)}(x_1) = W^{(1,0)}(x_1) - W^{(2,0)}(x_1).$$

The limit equations (2.2.38) are qualitatively different to the one given in Section 2.2.1 (see Equation (2.2.30)). As before, to remove the leading order discrepancy in the boundary conditions at the ends $x_1 = \pm l$ we set (similar to the Relationship (2.2.31))

$$\boxed{W^{(i,0)}(\pm l) = 0, \quad i = 1, 2.} \tag{2.2.39}$$

Consequently, the displacement jump $D^{(0)}(x_1)$ satisfies the following second-order differential equation

$$\boxed{\partial_1^2 D^{(0)}(x_1) - \frac{\mu(\mu_1 + \mu_2)}{\mu_1 \mu_2 H^2} D^{(0)}(x_1) = -\frac{1}{H} \int_{-H/2}^{H/2} \left(\frac{f_1(x_1, t)}{\mu_1} - \frac{f_2(x_1, t)}{\mu_2} \right) dt} \tag{2.2.40}$$

and the boundary conditions follow from (2.2.39)

$$\boxed{D^{(0)}(\pm l) = 0.} \tag{2.2.41}$$

2.2.4 Non-zero traction conditions

For the special case when $f_i = 0, \ i = 0, 1, 2$ and instead of homogeneous Neumann conditions on Γ_+, Γ_-, nonzero tractions are prescribed,

$$\begin{aligned} \frac{\partial}{\partial t_1} W^{(1,k)} &= \delta_{k2} p^{(1)}(x_1) \quad \text{on } \Gamma_+, \\ \frac{\partial}{\partial t_2} W^{(2,k)} &= \delta_{k2} p^{(2)}(x_1) \quad \text{on } \Gamma_-, \end{aligned}$$

the following system of ordinary differential equations for the leading terms is obtained,

$$\boxed{\begin{aligned} \partial_1^2 W^{(1,0)}(x_1) &= \frac{\mu}{\mu_1 h_0 h_1} \left\{ W^{(1,0)}(x_1) - W^{(2,0)}(x_1) \right\} - \frac{p^{(1)}(x_1)}{\mu_1 h_1}, \\ \partial_1^2 W^{(2,0)}(x_1) &= -\frac{\mu}{\mu_2 h_0 h_2} \left\{ W^{(1,0)}(x_1) - W^{(2,0)}(x_1) \right\} + \frac{p^{(2)}(x_1)}{\mu_2 h_2}. \end{aligned}} \tag{2.2.42}$$

Also, it is easily seen that the following operator is derived for the displacement jump,

$$\boxed{h_1 h_2 \partial_1^2 D^{(0)}(x_1) - \frac{\mu(\mu_1 h_1 + \mu_2 h_2)}{\mu_1 \mu_2 h_0} D^{(0)}(x_1) = -\left(\frac{p^{(1)}(x_1) h_2}{\mu_1} + \frac{p^{(2)}(x_1) h_1}{\mu_2}\right),}$$

where different thicknesses for the upper, middle and lower layers have been taken into account. As in the other cases, the boundary conditions (2.2.41) compensate for the error near the edges of the layered structure.

Equations (2.2.42) are the limit equation for the thin layered structure under the deformation of an anti-plane shear for the case when $\mu_0 = \epsilon^3 \mu$. These limit equations differ from the results available in literature. We can only specify the effective elastic modulus for the layered structure for the case when the functions $W^{(1,0)}$ and $W^{(2,0)}$ are equal.

2.2.5 Classification summary

A classification of limit equations obtained by varying the relation between the softness and the thinness of the imperfect interface that joins two elastic materials has been given.

For the case when $\mu_0 = \epsilon\mu$ (see Section 2.2.1), the limit equation (2.2.30) with the boundary conditions (2.2.31) for the leading term $W^{(0)}$ of the asymptotic solution (2.2.7) has been derived. The displacement jump has the order $O(\epsilon^2)$.

For the case when $\mu_0 = \epsilon^2 \mu$ (see Section 2.2.2), the analysis only changes a little. The unique difference is that the displacement jump has the order $O(\epsilon)$. The solution $W^{(0)}$ for all three layers is the same and satisfies the one-dimensional boundary value problem (2.2.30)–(2.2.31).

For the case when $\mu_0 = \epsilon^3 \mu$ (see Section 2.2.3) the displacement jump is of order $O(1)$ and satisfies the differential equation (2.2.40) and the boundary conditions (2.2.41). The limit equations (2.2.38) are qualitatively different from the Equation (2.2.30) in Section 2.2.1.

2.3 Boundary layer

In this section, as an illustrative example, the case when the right end conditions (2.2.6) are not homogeneous is considered. This analysis is given for the case when $\mu_0 = \epsilon^3 \mu$. Also we will take into account that there are non-zero tractions prescribed on the top and bottom of the three-layered structure

$$\frac{\partial}{\partial x_2} w^{(1)} = \epsilon p^{(1)}(x_1) \text{ on } \Gamma_+, \tag{2.3.1}$$

$$\frac{\partial}{\partial x_2} w^{(2)} = \epsilon p^{(2)}(x_1) \text{ on } \Gamma_-. \tag{2.3.2}$$

2.3.1 Formulation of the problem

To begin this analysis, let us assume that the left ends of the compound beam are under Dirichlet boundary conditions

$$w^{(i)}(-l, x_2) = \psi_-^{(i)}(\epsilon^{-1}x_2), i = 0, 1, 2, \tag{2.3.3}$$

and the right ends are assumed to be under homogeneous Dirichlet boundary conditions for the sake of simplicity, without loss of generality

$$w^{(i)}(l, x_2) = 0, i = 0, 1, 2. \tag{2.3.4}$$

Here the subindex "$-$" refers to the left end of the three-layered structure and "$-$" to the left end. The leading terms of the asymptotic series (2.2.7) $W^{(1,0)}$ and $W^{(2,0)}$ specified by the system of ordinary differential equations (2.2.38) do not necessarily satisfy the boundary conditions at the end $x_1 = -l$. That is,

$$W^{(i,0)}(-l) \neq \psi_-^{(i)}(\epsilon^{-1}x_2), i = 1, 2.$$

Hence, boundary layer solutions $v_-^{(i)}$ must be given to compensate for the discrepancy near the edges of the beam. This problem is one of the class of singularly perturbed boundary value problems. We will consider the left end of the layered structure. The error to be removed is given by

$$\psi_-^{(i)}(\epsilon^{-1}x_2) - W^{(i,0)}(-l), i = 0, 1, 2.$$

We introduce the scaled variables $\xi = (\frac{x_1+l}{\epsilon}, t_i)$, where t_1, t_2 are given by Relationship (2.1.2) and

$$t_0 = \epsilon^{-1}(x_2 + \epsilon(h/2 - h_2) - \epsilon^2 h_0/2).$$

Therefore the equation to be solved is given by

$$\Delta v_-^{(i)}(x, t_i) = 0, \text{ in } \Pi_i, i = 0, 1, 2, \tag{2.3.5}$$

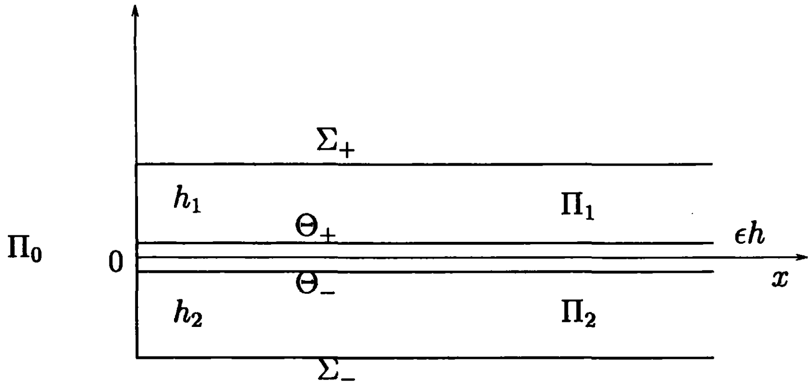

Figure 2.2: The semi-infinite strip.

where $x > 0$ and $t_i, i = 0, 1, 2$ are specified as follows,

$$t_1 \in \left[-\frac{h_1}{2}, \frac{h_1}{2}\right],$$

$$t_2 \in \left[-\frac{h_2}{2}, \frac{h_2}{2}\right],$$

$$t_0 \in \left[-\frac{\epsilon h_0}{2}, \frac{\epsilon h_0}{2}\right]$$

and the regions $\Pi_i, i = 0, 1, 2$ (see Figure 2.2) are given by

$$\Pi_1 = \left\{x > 0\,,\, -\frac{h_1}{2} < t_1 < \frac{h_1}{2}\right\},$$

$$\Pi_2 = \left\{x > 0\,,\, -\frac{h_2}{2} < t_2 < \frac{h_2}{2}\right\},$$

$$\Pi_0 = \left\{x > 0\,,\, -\epsilon\frac{h_0}{2} < t_0 < \epsilon\frac{h_0}{2}\right\}.$$

The following free-traction conditions are prescribed

$$\frac{\partial}{\partial t_1} v_-^{(1)}(x, t_1) = 0 \quad \text{on } \Sigma_+, \tag{2.3.6}$$

$$\frac{\partial}{\partial t_2} v_-^{(2)}(x, t_2) = 0 \quad \text{on } \Sigma_-, \tag{2.3.7}$$

where the upper and lower surfaces of the compound region are specified as follows

$$\Sigma_+ = \left\{ x > 0 \, , \, t_1 = \frac{h_1}{2} \right\},$$
$$\Sigma_- = \left\{ x > 0 \, , \, t_2 = -\frac{h_2}{2} \right\}.$$

The Dirichlet end conditions are given by

$$v_-^{(i)}(0, t_i) = \psi_-^{(i)}(t_i) - W^{(i,0)}(-l), i = 0, 1, 2. \tag{2.3.8}$$

Here it is assumed that the materials are perfectly bonded at the interfaces Θ_+ and Θ_-,

$$v_-^{(0)}\left(x, \frac{\epsilon h_0}{2}\right) = v_-^{(1)}\left(x, -\frac{h_1}{2}\right), \tag{2.3.9}$$
$$v_-^{(0)}\left(x, -\frac{\epsilon h_0}{2}\right) = v_-^{(2)}\left(x, \frac{h_2}{2}\right), \tag{2.3.10}$$

$$\mu_1 \frac{\partial}{\partial t_1} v_-^{(1)}(x, t_1) = \mu_0 \frac{\partial}{\partial t_0} v_-^{(0)}(x, t_0) \text{ on } \Theta_+, \tag{2.3.11}$$
$$\mu_2 \frac{\partial}{\partial t_2} v_-^{(2)}(x, t_2) = \mu_0 \frac{\partial}{\partial t_0} v_-^{(0)}(x, t_0) \text{ on } \Theta_-, \tag{2.3.12}$$

where the interfaces are given by

$$\Theta_+ = \left\{ x > 0 \, , t_1 = -\frac{h_1}{2}, \; t_0 = \epsilon \frac{h_0}{2} \right\},$$
$$\Theta_- = \left\{ x > 0 \, , t_2 = \frac{h_2}{2}, \; t_0 = -\epsilon \frac{h_0}{2} \right\}.$$

2.3.2 Solution by separation of variables

To simplify our analysis we will assume that $h_1 = h_2 \equiv h^*$ and that the materials of the upper and lower layers have the same Poisson ratio, $\mu_1 = \mu_2 \equiv \mu^*$. Using the separation of variables method, we write the solution of the equations (2.3.5) as

$$v_-^{(i)}(x, t_i) = X^{(i)}(x) Y^{(i)}(t_i), i = 0, 1, 2. \tag{2.3.13}$$

When we put a function of this form into (2.3.5), the partial derivatives in the differential equation appear as ordinary derivatives on the functions $X^{(i)}$ and $Y^{(i)}$; in other words the equations (2.3.5) become

$$X^{(i)\prime\prime}(x)Y^{(i)}(t_i) + X^{(i)}(x)Y^{(i)\prime\prime}(t_i) = 0. \tag{2.3.14}$$

At any point (x, t_i) at which $v_-^{(i)}(x, t_i)$ are non-zero we can divide the Equation (2.3.14) by $v_-^{(i)}(x, t_i)$ and rearrange to get

$$\frac{X^{(i)\prime\prime}(x)}{X^{(i)}(x)} = -\frac{Y^{(i)\prime\prime}(t_i)}{Y^{(i)}(t_i)} = \left(\frac{2\lambda^{(i)}}{H_i}\right)^2 \tag{2.3.15}$$

where

$$\left(\frac{2\lambda^{(i)}}{H_i}\right)^2 = \chi_i^2 > 0$$

is a constant written in this way for convenience and the quantities H_i are given as follows

$$H_1 = H_2 \equiv h^*, \; H_0 = \epsilon h_0. \tag{2.3.16}$$

The Equation (2.3.15) gives the following pair of equations

$$X^{(i)\prime\prime}(x) - \chi_i^2 X^{(i)}(x) = 0, \tag{2.3.17}$$
$$Y^{(i)\prime\prime}(t_i) + \chi_i^2 Y^{(i)}(t_i) = 0. \tag{2.3.18}$$

Using the following condition

$$v_-^{(i)}(x, t_i) \to 0 \text{ as } x \to \infty,$$

it can be seen that the solutions of the Equations (2.3.17) and (2.3.18) are given by

$$\begin{aligned} X^{(i)}(x) &= e^{-\chi_i x}, \\ Y^{(i)}(t_i) &= \alpha^{(i)} \cos(\chi_i t_i) + \beta^{(i)} \sin(\chi_i t_i). \end{aligned} \tag{2.3.19}$$

Using Equations (2.3.19), the solution (2.3.13) can be written as

$$\boxed{v_-^{(i)}(x, t_i) = e^{-\chi_i x}(\alpha^{(i)} \cos(\chi_i t_i) + \beta^{(i)} \sin(\chi_i t_i)), i = 0, 1, 2,} \tag{2.3.20}$$

where $\alpha^{(i)}, \beta^{(i)}, i = 0, 1, 2$ are constants to be determined using the Dirichlet end conditions (2.3.8).

Using the continuity of $v_-^{(i)}$ at the interface given by Relationships (2.3.9) and (2.3.10) we can see that

$$\chi_1 = \chi_2 = \chi_0$$

and in particular, by (2.3.16) we get

$$\boxed{\lambda^{(1)} = \lambda^{(2)} \equiv \lambda^{(*)}.}$$

2.3.3 Decay rate of the boundary layer solution

General anti-plane shear problems for linear and non-linear solid mechanics have been studied by Horgan (1995). The problem of analysing the end effects for a sandwich structure with perfect and imperfect bonding has been studied by Baxter and Horgan (1995) and Baxter and Horgan (1997). Following their notation we introduce the quantity k as follows,

$$\frac{2\lambda^{(*)}}{h^*} = \frac{2\lambda^{(0)}}{\epsilon h_0} = \frac{2\lambda}{2h^* + \epsilon h_0} \equiv k.$$

The exponential decay rates $\frac{2\lambda^{(i)}}{H_i}$ can be compared with that for a homogeneous strip of *weighted* total half-width $(h^* + \frac{\epsilon h_0}{2})$. Defining a non-dimensional weighted volume fraction as

$$f = \frac{2h^*}{2h^* + \epsilon h_0}, \tag{2.3.21}$$

we express $\lambda^{(*)}$ and $\lambda^{(0)}$ in terms of λ and f as follows

$$\lambda^{(*)} = \frac{f\lambda}{2}; \;\; \lambda^{(0)} = \lambda(1 - f).$$

Using the boundary conditions (2.3.6) and (2.3.7) and continuity conditions (2.3.9)–(2.3.12), the following homogeneous system of algebraic equations is formulated for the unknown coefficients $\alpha^{(i)}$ and $\beta^{(i)}$,

$$\boldsymbol{F}\Xi = \boldsymbol{0},$$

where the coefficients of the 6×6 matrix $\boldsymbol{F}$ are given by

$$\begin{aligned}
F_{11} &= \cos(\tfrac{f\lambda}{2}), & F_{12} &= -\sin(\tfrac{f\lambda}{2}),\\
F_{13} &= -\cos(\lambda(1-f)), & F_{14} &= -\sin(\lambda(1-f),\\
F_{23} &= \cos(\lambda(1-f)), & F_{24} &= -\sin(\lambda(1-f)),\\
F_{25} &= -\cos(\tfrac{f\lambda}{2}), & F_{26} &= -\sin(\tfrac{f\lambda}{2}),\\
F_{31} &= \tfrac{f\lambda}{h^*}\sin(\tfrac{f\lambda}{2}), & F_{32} &= \tfrac{f\lambda}{h^*}\cos(\tfrac{f\lambda}{2}),\\
F_{33} &= \tfrac{2\mu_0\lambda(1-f)}{\mu_1\epsilon h_0}\sin(\lambda(1-f)), & F_{34} &= -\tfrac{2\mu_0\lambda(1-f)}{\mu_1\epsilon h_0}\cos(\lambda(1-f)),\\
F_{43} &= \tfrac{2\mu_0\lambda(1-f)}{\mu_2\epsilon h_0}\sin(\lambda(1-f)), & F_{44} &= \tfrac{2\mu_0\lambda(1-f)}{\mu_2\epsilon h_0}\cos(\lambda(1-f)),\\
F_{45} &= \tfrac{f\lambda}{h^*}\sin(\tfrac{f\lambda}{2}), & F_{46} &= -\tfrac{f\lambda}{h^*}\cos(\tfrac{f\lambda}{2}),\\
F_{51} &= \tfrac{f\lambda}{h^*}\sin(\tfrac{f\lambda}{2}), & F_{52} &= -\tfrac{f\lambda}{h^*}\cos(\tfrac{f\lambda}{2}),\\
F_{65} &= \tfrac{f\lambda}{h^*}\sin(\tfrac{f\lambda}{2}), & F_{66} &= \tfrac{f\lambda}{h^*}\cos(\tfrac{f\lambda}{2}),
\end{aligned}$$

and the remaining terms are zero; the vector Ξ is given by

$$\Xi = \left(\alpha_j^{(0)}, \beta_j^{(0)}, \alpha_j^{(1)}, \beta_j^{(1)}, \alpha_j^{(2)}, \beta_j^{(2)}\right)^T.$$

We seek non-trivial solutions Ξ, when the determinant of $\boldsymbol{F}$ vanishes. Taking into account the simplifications for this special case where the upper and lower layers are symmetric, it is easy to find that the characteristic polynomial can be expressed by

$$\begin{aligned}
&-\left(\frac{\mu^*}{\mu_0}\right)^2 \sin^2(f\lambda)\sin(2\lambda(1-f)) + \cos^2(f\lambda)\sin(2\lambda(1-f))\\
&+\frac{2\mu^*}{\mu_0}\cos(f\lambda)\cos(2\lambda(1-f))\sin(f\lambda) = 0.
\end{aligned} \tag{2.3.22}$$

We assume that ϵ is sufficiently small and $f \neq \frac{1}{2}$ (see (2.3.21)). Following Baxter and Horgan (1997) one writes (2.3.22) more simply as

$$\left(\cot(f\lambda)\cot(\lambda(1-f)) - \frac{\mu^*}{\mu_0}\right)\left(\cot(f\lambda)\tan(\lambda(1-f)) + \frac{\mu^*}{\mu_0}\right) = 0. \tag{2.3.23}$$

It has been proved by Baxter and Horgan (1997) that the smallest positive root of Equation (2.3.23) arises from the first factor. So, we find this root λ from the following relationship

$$\boxed{\cot(f\lambda)\cot(\lambda(1-f)) - \frac{\mu^*}{\mu_0} = 0.} \tag{2.3.24}$$

Given that the middle layer is softer than the upper and lower layers, we substitute μ_0 with $\epsilon^3\mu$, where μ has the same order of magnitude as μ^*. Thus, the Equation (2.3.24) can be written as

$$\cot(f\lambda)\cot(\lambda(1-f)) = \epsilon^{-3}\frac{\mu^*}{\mu}. \tag{2.3.25}$$

The quantity $1 - f$ is estimated as follows,

$$1 - f = \frac{\epsilon h_0}{2h^* + \epsilon h_0} \sim \frac{\epsilon h_0}{2h^*},$$

therefore

$$f = 1 - \frac{\epsilon h_0}{2h^*} + \cdots .$$

Also we notice that

$$\cot\left(\frac{\epsilon h_0}{2h^*}\lambda\right) \sim \frac{2h^*}{\epsilon h_0 \lambda},$$

so we find that the relationship (2.3.25) can be approximated as

$$\frac{1}{\lambda}\cot(\lambda\frac{2h^* - \epsilon h_0}{2h^*}) \sim \frac{1}{\lambda}\cot(\lambda) \sim \epsilon^{-2}\frac{\mu^* h_0}{2\mu h^*}.$$

For small values of λ, we have the following estimate

$$\cot(\lambda) \sim \frac{1}{\lambda},$$

so, finally we can give an approximation for λ

$$\boxed{\lambda \sim \epsilon\left(\frac{2\mu h^*}{\mu^* h_0}\right)^{1/2}.} \tag{2.3.26}$$

This approximation of λ for (2.3.23) clearly includes the small parameter ϵ.

Hence the decay rate is of order $O(1)$ since $x = \frac{x_1+l}{\epsilon}$. Thus k predicts a very slow decay rate in this case in contrast with the well-known behaviour for a layered structure with perfect bonding or for a homogeneous beam. If $\mu_0 = \mu^*$, so that the strip is homogeneous, the smallest positive root is

$$\lambda = \frac{\pi}{2},$$

so that w decays as

$$\exp(-kx_1), \quad k = \frac{\pi}{h},$$

where h is the beam width (see Horgan and Knowles (1983) Horgan (1989) and Horgan and Simmonds (1994)).

2.3.4 The end conditions

A complete solution to Equations (2.3.5), subject to prescribed boundary conditions at $x = 0$ would involve an infinite series of eigenfunctions (including the constant solution) of the form (2.3.20) with the following representation

$$\boxed{v_-^{(i)}(x,t_i) = \sum_{j=0}^{\infty}\left\{\exp(-\tfrac{2\lambda_j^{(i)}}{H_i}x)\left[\alpha_j^{(i)}\cos(\tfrac{2\lambda_j^{(i)}}{H_i}t_i) + \beta_j^{(i)}\sin(\tfrac{2\lambda_j^{(i)}}{H_i}t_i)\right]\right\},} \tag{2.3.27}$$

for $i = 0, 1, 2$. Here the coefficients $\alpha_j^{(i)}, \beta_j^{(i)}, i = 0, 1, 2; j \geq 1$ are the same as in Baxter and Horgan (1997).

To find the value of the coefficient $\alpha_0^{(i)}$ in each layer, which is desirable to vanish (in other words, when $\chi_i = 0$), one has to use the end conditions (2.3.8). Evaluating the solution of the problem in Equations (2.3.27) at the left end, it follows that

$$\begin{aligned} v_-^{(i)}(0,t_i) &= \alpha_0^{(i)} + \sum_{j=1}^{\infty}\left[\alpha_j^{(i)}\cos\left(\frac{2\lambda_j^{(i)}t_i}{H_i}\right) + \beta_j^{(i)}\sin\left(\frac{2\lambda_j^{(i)}t_i}{H_i}\right)\right] \\ &= \psi_-^{(i)}(t_i) - W^{(i,0)}(-l). \end{aligned}$$

Integrating with respect to the scaled cross-section variable t_i in each layer, one finds that

$$\int(\psi_-^{(i)}(t_i) - W^{(i,0)}(-l))dt_i = \int \alpha_0^{(i)}dt_i.$$

The functions $v_-^{(i)}$ of the boundary layer type vanish at infinity if and only if

$$\boxed{W^{(i,0)}(-l) = \frac{1}{H_i}\int \psi_-^{(i)}(t_i)dt_i,} \tag{2.3.28}$$

that constitutes the left end conditions of the thin layered structure under normal and longitudinal loading.

2.3.5 Effect at the right end

Since the function $v_-^{(i)}$ introduces a finite discrepancy in the boundary condition at the right end we have to construct another function at the right end to compensate for the error left by $v_-^{(i)}$. We construct a function at the left end with the following structure

$$\boxed{V_-^{(i)}(x_1, t_i) = \sum_{j=1}^{\infty} c_j \exp(-\tfrac{2\hat{\lambda}_j^{(i)}}{H_i}(x_1 + l))Y_j^{(i)}(t_i, \epsilon),} \tag{2.3.29}$$

where $\lambda_j^{(i)} = \epsilon\hat{\lambda}_j^{(i)}$ is the normalised exponent and c_j are constants to be specified. It is emphasised that $\int Y_j^{(i)}dt_i = 0$ when we integrate over the cross-section.

The analysis for the right end of the thin layered structure involves functions $V_+^{(i)}, i = 0, 1, 2$ with the following structure:

$$\boxed{V_+^{(i)}(x_1, t_i) = -\sum_{j=1}^{\infty} d_j \exp(-\tfrac{2\hat{\lambda}_j^{(i)}}{H_i}(l - x_1))Y_j^{(i)}(t_i, \epsilon),} \tag{2.3.30}$$

The functions $V_-^{(i)}$ and $V_+^{(i)}$ can be considered as a result of a multiple reflection from the ends of the layered structure.

We look for the combination $V_-^{(i)} + V_+^{(i)}$, satisfying the following relationships at the ends

$$\begin{aligned}(V_-^{(i)} + V_+^{(i)})|_{x_1=-l} &= \sum_{j\geq 1} Y_j^{(i)}(t_i)\\ (V_-^{(i)} + V_+^{(i)})|_{x_1=l} &= 0.\end{aligned} \tag{2.3.31}$$

After evaluating the functions (2.3.29) and (2.3.30) at the ends and substituting them into the system (2.3.31) we get a system for the constants c_j and d_j

$$c_j - d_j \exp(-\frac{4\hat{\lambda}_j^{(i)} l}{H_i}) = 1,$$
$$c_j \exp(-\frac{4\hat{\lambda}_j^{(i)} l}{H_i}) - d_j = 0, j \geq 1.$$

The solution for the constants gives

$$c_j = \frac{1}{1 - \exp(-\frac{8\hat{\lambda}_j^{(i)} l}{H_i})}; \quad d_j = \frac{\exp(-\frac{4\hat{\lambda}_j^{(i)} l}{H_i})}{1 - \exp(-\frac{8\hat{\lambda}_j^{(i)} l}{H_i})}. \tag{2.3.32}$$

Analogously, one can derive the following condition at the right end by using the homogeneous Dirichlet boundary condition (2.3.4),

$$\boxed{W^{(i,0)}(+l) = 0.} \tag{2.3.33}$$

2.3.6 Discrepancy in equations and boundary conditions

The function $w^{(i)}$ from the Relationship (2.2.2) with boundary and end conditions (2.3.1)–(2.3.3) can be represented in the form

$$\begin{aligned} w^{(i)} \sim\ & W^{(i,0)}(x_1) + \epsilon W^{(i,1)}(x_1) + \epsilon^2 W^{(i,2)}(x_1, t_i) \\ & + V_-^{(i)}(x, t_i) + V_+^{(i)}(x, t_i) + \tilde{W}^{(i)}(\boldsymbol{x}, \epsilon), \qquad i = 0, 1, 2. \end{aligned} \tag{2.3.34}$$

In the last Relationship, $V_-^{(i)} + V_+^{(i)}, i = 0, 1, 2$ compensate for the discrepancy left by $W^{(1,0)}(x_1)$ and $W^{(2,0)}(x_1)$ at the left end.

The problem (2.2.42),(2.3.28)–(2.3.33) describes completely the leading order terms of the displacement field within a thin rectangular layered structure including a soft and thin middle layer subject to the state of deformation under an anti-plane shear.

The remainders $\tilde{W}^{(i)}(\boldsymbol{x}, \epsilon), i = 0, 1, 2$ in Relationship (2.3.34) solve the following equations

$$\begin{aligned} -\Delta \tilde{W}^{(i)}(\boldsymbol{x}, \epsilon) = \epsilon^2 \{ & \partial_1^2 W^{(i,0)}(x_1) + \epsilon \partial_1^2 W^{(i,1)}(x_1) \\ & + \Delta W^{(i,2)}(x_1, t_i) \}, \end{aligned} \tag{2.3.35}$$

and satisfy the boundary conditions

$$
\begin{aligned}
\partial_{t_1}\tilde{W}^{(1)}(\boldsymbol{x},\epsilon) &= 0 && \text{on } \Gamma_+,\\
\mu_1\partial_{t_1}\tilde{W}^{(1)}(\boldsymbol{x},\epsilon) &= \epsilon^2(\epsilon^3\mu\partial_{t_0}W^{(0,2)}(x,t_0) - \mu_1\partial_{t_1}W^{(1,2)}) && \\
&\quad +\epsilon^3\mu\partial_{t_0}\tilde{W}^{(0)} && \text{on } S_+,
\end{aligned}
\tag{2.3.36}
$$

for the upper layer,

$$
\begin{aligned}
\mu_1\partial_{t_2}\tilde{W}^{(2)}(\boldsymbol{x},\epsilon) &= \epsilon^2(\epsilon^3\mu\partial_{t_0}W^{(0,2)}(x,t_0) - \mu_2\partial_{t_2}W^{(2,2)}) && \\
&\quad +\epsilon^3\mu\partial_{t_0}\tilde{W}^{(0)} && \text{on } S_-,\\
\partial_{t_2}\tilde{W}^{(2)}(\boldsymbol{x},\epsilon) &= 0 && \text{on } \Gamma_-,
\end{aligned}
\tag{2.3.37}
$$

for the lower layer and

$$
\begin{aligned}
\tilde{W}^{(1)}(x,-h_1/2) &= \tilde{W}^{(0)}(x,\epsilon h_0/2) \quad \text{on } S_+,\\
\tilde{W}^{(2)}(x,h_2/2) &= \tilde{W}^{(0)}(x,-\epsilon h_0/2) \quad \text{on } S_-,
\end{aligned}
\tag{2.3.38}
$$

for the middle layer. The following end conditions are also satisfied

$$
\tilde{W}^{(i)}(\pm l) = -\epsilon(W^{(i,1)}(\pm l) + \epsilon W^{(i,2)}(\pm l, t_i)). \tag{2.3.39}
$$

The expansion (2.3.34) generates a discrepancy of order $O(\epsilon)$ (see Relationship (2.3.39)) at the end conditions (2.3.3) and of order $O(\epsilon^2)$ (see Relationship (2.3.35)) in the Equations (2.2.2).

2.4 Plane strain

In this section a boundary value problem for the Lamé operator that corresponds to the case of plane strain of the layered structure is studied. The asymptotic analysis shows that different values of the normalised Young's modulus of the middle layer is important.

The elastic materials of the regions $\Omega_i, i = 0,1,2$ are characterised by the Young's moduli E_i and by the values ν_i of the Poisson ratio, $i = 0,1,2$. By $\lambda_i, \mu_i, i = 0,1,2$ we denote the Lamé constants of the elastic materials

$$
\lambda_i = \frac{E_i\nu_i}{(1+\nu_i)(1-2\nu_i)}, \mu_i = \frac{E_i}{2(1+\nu_i)}. \tag{2.4.1}
$$

Following the idea of Klarbring and Movchan (1998), the Young's modulus of the middle layer is normalised in the following way

$$E_0 = \epsilon^m E,$$

where m is positive integer and E has the same order of magnitude as E_1 and E_2. It is considered that the Poisson ratio has the same order of magnitude for the three layers. Thus, the normalised Lamé constants related to the middle layer are given by

$$\lambda = \frac{E\nu_0}{(1+\nu_0)(1-2\nu_0)}, \mu = \frac{E}{2(1+\nu_0)}. \tag{2.4.2}$$

2.4.1 Two illustrative cases

It can be shown that the most interesting case occurs when $m = 3$. As a literally illustrative demonstration, we will study the cases $m = 1$ and $m = 2$. For simplicity, we will consider a layered structure with an imperfect interface where upper and lower layers are made of the same material and have the same thickness. Here, the case when the transverse loading is absent, is studied. It turns out that the leading order term for the longitudinal components of the displacement field are continuous across the middle layer. Hence this leading order term satisfies the equation given by Relationship (1.4.30).

2.4.2 Equations and Boundary Conditions

Here, the state of deformation of the layered structure under a plane strain is considered. It is assumed that the vectors of the displacement field $\boldsymbol{u}^{(i)}$ satisfy the homogeneous Lamé systems

$$\mu_i \nabla^2 \boldsymbol{u}^{(i)}(\boldsymbol{x}) + (\lambda_i + \mu_i)\nabla\nabla \cdot \boldsymbol{u}^{(i)}(\boldsymbol{x}) = 0, \ \boldsymbol{x} \in \Omega_i, \ i = 0, 1, 2. \tag{2.4.3}$$

For the surfaces of the compound region Ω_ϵ we prescribe tractions:

$$\mu_1 \left(\frac{\partial u_2^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_2} \right) = \epsilon p_1^{(1)}, \ (2\mu_1 + \lambda_1)\frac{\partial u_2^{(1)}}{\partial x_2} + \lambda_1 \frac{\partial u_1^{(1)}}{\partial x_1} = 0 \text{ on } \Gamma_+, \tag{2.4.4}$$

$$\mu_2 \left(\frac{\partial u_2^{(2)}}{\partial x_1} + \frac{\partial u_1^{(2)}}{\partial x_2} \right) = \epsilon p_1^{(2)}, \ (2\mu_2 + \lambda_2)\frac{\partial u_2^{(2)}}{\partial x_2} + \lambda_2 \frac{\partial u_1^{(2)}}{\partial x_1} = 0 \text{ on } \Gamma_-. \tag{2.4.5}$$

The conditions of the ideal interface contact are specified:

$$\mu_1\left(\frac{\partial u_2^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_2}\right)=\mu_0\left(\frac{\partial u_2^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_2}\right), \tag{2.4.6}$$

$$(2\mu_1+\lambda_1)\frac{\partial u_2^{(1)}}{\partial x_2}+\lambda_1\frac{\partial u_1^{(1)}}{\partial x_1}=(2\mu_0+\lambda_0)\frac{\partial u_2^{(0)}}{\partial x_2}+\lambda_0\frac{\partial u_1^{(0)}}{\partial x_1}, \tag{2.4.7}$$

$$\boldsymbol{u}^{(1)}(\boldsymbol{x})=\boldsymbol{u}^{(0)}(\boldsymbol{x}),\ \ \boldsymbol{x}\in S_+ \tag{2.4.8}$$

and

$$\mu_2\left(\frac{\partial u_2^{(2)}}{\partial x_1}+\frac{\partial u_1^{(2)}}{\partial x_2}\right)=\mu_0\left(\frac{\partial u_2^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_2}\right), \tag{2.4.9}$$

$$(2\mu_2+\lambda_2)\frac{\partial u_2^{(2)}}{\partial x_2}+\lambda_2\frac{\partial u_1^{(2)}}{\partial x_1}=(2\mu_0+\lambda_0)\frac{\partial u_2^{(0)}}{\partial x_2}+\lambda_0\frac{\partial u_1^{(0)}}{\partial x_1}, \tag{2.4.10}$$

$$\boldsymbol{u}^{(2)}(\boldsymbol{x})=\boldsymbol{u}^{(0)}(\boldsymbol{x}),\ \ \boldsymbol{x}\in S_-. \tag{2.4.11}$$

The ends of a thin compound beam are assumed to be fixed:

$$\boldsymbol{u}^{(i)}(\pm l,x_2)=\boldsymbol{0},\ i=0,1,2.$$

2.4.3 Asymptotic Procedure

We assume that the displacement is approximated by

$$\boldsymbol{u}^{(i)}(\boldsymbol{x})\sim\sum_{k=0}^{N}\epsilon^k\boldsymbol{u}^{(i,k)}(x_1,t_i),\ i=0,1,2. \tag{2.4.12}$$

In the text below we consider k running up to 4.

Putting the last series (2.4.12) into system (2.4.3) and boundary conditions (2.4.4)–(2.4.11) and equating the coefficients of like powers of ϵ, the following system of recurrence relations is obtained valid on the cross-section,

$$\mu_i\partial_{t_i}^2u_1^{(i,k)}+(\lambda_i+\mu_i)\partial_{t_i1}^2u_2^{(i,k-1)}+(\lambda_i+2\mu_i)\partial_1^2u_1^{(i,k-2)}=0, \tag{2.4.13}$$

$$\mu_i\partial_1^2u_2^{(i,k-2)}+(\lambda_i+\mu_i)\partial_{t_i1}^2u_1^{(i,k-1)}+(\lambda_i+2\mu_i)\partial_{t_i}^2u_2^{(i,k)}=0 \text{ in } \Omega_i, \tag{2.4.14}$$

for the upper and lower layers where $i=1,2$ and

$$\mu\partial_{t_0}^2u_1^{(0,k)}+(\lambda+\mu)\partial_{t_01}^2u_2^{(0,k-2)}+(\lambda+2\mu)\partial_1^2u_1^{(0,k-4)}=0, \tag{2.4.15}$$

$$\mu\partial_1^2u_2^{(0,k-4)}+(\lambda+\mu)\partial_{t_01}^2u_1^{(0,k-2)}+(\lambda+2\mu)\partial_{t_0}^2u_2^{(0,k)}=0, \text{in } \Omega_0, \tag{2.4.16}$$

for the middle layer.

On the interface boundary

$$\begin{aligned} \mu_1(\partial_{t_1} u_1^{(1,k)} + \partial_1 u_2^{(1,k-1)}) &= \mu(\partial_{t_0} u_1^{(0,k+1-m)} + \partial_1 u_2^{(0,k-1-m)}), \\ \lambda_1 \partial_1 u_1^{(1,k-1)} + (\lambda_1 + 2\mu_1)\partial_{t_1} u_2^{(1,k)} &= (\lambda + 2\mu)\partial_{t_0} u_2^{(0,k+1-m)} + \lambda \partial_1 u_1^{(0,k-1-m)}, \\ \boldsymbol{u}^{(0,k)} &= \boldsymbol{u}^{(1,k)} \text{ on } S_+ \end{aligned} \tag{2.4.17}$$

and

$$\begin{aligned} \mu_2(\partial_{t_2} u_1^{(2,k)} + \partial_1 u_2^{(2,k-1)}) &= \mu(\partial_{t_0} u_1^{(0,k+1-m)} + \partial_1 u_2^{(0,k-1-m)}), \\ \lambda_2 \partial_1 u_1^{(2,k-1)} + (\lambda_2 + 2\mu_2)\partial_{t_2} u_2^{(2,k)} &= (\lambda + 2\mu)\partial_{t_0} u_2^{(0,k+1-m)} + \lambda \partial_1 u_1^{(0,k-1-m)}, \\ \boldsymbol{u}^{(0,k)} &= \boldsymbol{u}^{(2,k)} \text{ on } S_-. \end{aligned} \tag{2.4.18}$$

On the upper and lower surfaces:

$$\mu_1(\partial_{t_1} u_1^{(1,k)} + \partial_1 u_2^{(1,k-1)}) = \delta_{k2} p_1^{(1)}, \tag{2.4.19}$$

$$\lambda_1 \partial_1 u_1^{(1,k-1)} + (\lambda_1 + 2\mu_1)\partial_{t_1} u_2^{(1,k)} = 0 \text{ on } \Gamma_+ \tag{2.4.20}$$

and

$$\mu_2(\partial_{t_2} u_1^{(2,k)} + \partial_1 u_2^{(2,k-1)}) = \delta_{k2} p_1^{(2)}, \tag{2.4.21}$$

$$\lambda_2 \partial_1 u_1^{(2,k-1)} + (\lambda_2 + 2\mu_2)\partial_{t_2} u_2^{(2,k)} = 0 \text{ on } \Gamma_-. \tag{2.4.22}$$

We have the recurrent sequence of the Neumann boundary value problems on the cross-section for ordinary differential equations in the upper and lower layers (the longitudinal variable x_1 is included as a parameter) and the sequence of Dirichlet boundary value problems for the middle layer.

Neumann boundary value problems do not generally have a solution, unless they satisfy certain solvability conditions. The procedure for finding a solution for each Neumann BVP in each step of the recurrence systems (2.4.13) and (2.4.14), with the boundary conditions (2.4.17)–(2.4.21), will give a system of ordinary differential equations for the components of the leading order term of the asymptotic approximation. This system is precisely the set of solvability conditions of the Neumann BVPs.

2.4.4 Limit equations when $E_0 = \epsilon E$

In order to obtain a set of differential equations that constitute a well-posed system including the leading order, we investigate just five terms of the series

(2.4.12).

Step 1 If $m = 1$, in Equations (2.4.13) and (2.4.17)–(2.4.22), it is seen that for $k = 0$ the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_1^{(i,0)} &= 0, && \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_1^{(i,0)} &= 0, && \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_1^{(i,0)} &= \frac{\mu}{\mu_i} \partial_{t_0} u_1^{(0,0)}, && \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{2.4.23}$$

It follows that the solvability conditions for the BVPs (2.4.23) are given by

$$\partial_{t_0} u_1^{+(0,0)} = \partial_{t_0} u_1^{-(0,0)} = 0, \tag{2.4.24}$$

where the evaluation at the interfaces S_+ and S_- has been made according to the notation (2.2.15). Moreover, with the boundary conditions, the first terms of the approximation (2.4.12), for both the upper and lower layers, are found to be functions of x_1 only,

$$u_1^{(i,0)} = u_1^{(i,0)}(x_1). \tag{2.4.25}$$

For the middle layer, Equations (2.4.15), (2.4.17) and (2.4.18) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,0)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_1^{(1,0)}, && \text{on } S_+, \\ u_1^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_1^{(2,0)}, && \text{on } S_-. \end{aligned} \tag{2.4.26}$$

Therefore, using Relationship (2.4.24), it follows that

$$u_1^{(0,0)}(x_1) = u_1^{(1,0)}(x_1) = u_1^{(2,0)}(x_1) \equiv u_1^{(0)}(x_1). \tag{2.4.27}$$

When $k = 0$, Equations (2.4.14) and (2.4.17)–(2.4.22) give the BVPs on the cross-section for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_2^{(i,0)} &= 0, && \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_2^{(i,0)} &= 0, && \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_2^{(i,0)} &= \frac{\lambda + 2\mu}{\lambda_i + 2\mu_i} \partial_{t_0} u_2^{(0,0)}, && \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{2.4.28}$$

It follows that the solvability conditions for the BVP (2.4.28) are given by

$$\partial_{t_0} u_2^{+(0,0)} = \partial_{t_0} u_2^{-(0,0)} = 0. \tag{2.4.29}$$

Thus using relationship (2.4.29), the first terms of the approximation (2.4.12), for both the upper and lower layers, are found to be functions of x_1 only,

$$u_2^{(i,0)} = u_2^{(i,0)}(x_1). \tag{2.4.30}$$

For the middle layer, Equations (2.4.16)–(2.4.18) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_2^{(0,0)} &= 0, && \text{in } \Omega_0, \\ u_2^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_2^{(1,0)}, && \text{on } S_+, \\ u_2^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_2^{-(2,0)}, && \text{on } S_-. \end{aligned} \tag{2.4.31}$$

Therefore, using Relationship (2.4.29), it follows that

$$u_2^{(0,0)}(x_1) = u_2^{(1,0)}(x_1) = u_2^{(2,0)}(x_1) \equiv u_2^{(0)}(x_1). \tag{2.4.32}$$

The Relationships (2.4.27) and (2.4.32) establish that the leading order terms are continuous across the thin and soft middle layer.

Step 2 For k=1, Equations (2.4.14) and (2.4.17)–(2.4.22) satisfy the following Neumann BVP on the cross-section for the upper layer,

$$\begin{aligned} \partial_{t_1}^2 u_2^{(1,1)} &= 0, && \text{in } \Omega_1, \\ \partial_{t_1} u_2^{(1,1)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(0)}(x_1), && \text{on } \Gamma_+, \\ \partial_{t_1} u_2^{(1,1)} &= \frac{\lambda + 2\mu}{\lambda_1 + 2\mu_1}\partial_{t_0} u_2^{(0,1)} - \frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(0)}(x_1), && \text{on } S_+. \end{aligned} \tag{2.4.33}$$

The solvability condition for the Neumann BVP (2.4.33) is given by

$$\partial_{t_0} u_2^{+(0,1)} = 0. \tag{2.4.34}$$

Similarly for the lower layer, the following Neumann BVP on the cross-section is satisfied

$$\begin{aligned}
&\partial^2_{t_2} u_2^{(2,1)} = 0, && \text{in } \Omega_2, \\
&\partial_{t_2} u_2^{(2,1)} = -\frac{\lambda_2}{\lambda_2+2\mu_2}\partial_1 u_1^{(0)}(x_1), && \text{on } \Gamma_-(i=2), \\
&\partial_{t_2} u_2^{(2,1)} = \frac{\lambda+2\mu}{\lambda_2+2\mu_2}\partial_{t_0} u_2^{(0,1)} - \frac{\lambda_2}{\lambda_2+2\mu_2}\partial_1 u_1^{(0)}(x_1), && \text{on } S_-(i=2).
\end{aligned} \tag{2.4.35}$$

The solvability conditions for the Neumann BVP (2.4.35) is given by

$$\partial_{t_0} u_2^{-(0,1)} = 0. \tag{2.4.36}$$

The solution for $u_2^{(i,1)}, i = 1, 2$ gives

$$u_2^{(i,1)} = -\frac{\lambda_i t_i}{\lambda_i + 2\mu_i}\partial_1 u_1^{(0)}(x_1) + g^{(i)}(x_1), \tag{2.4.37}$$

where the functions $g^{(i)}(x_1)$ are unknown. For the middle layer, Equations (2.4.16)–(2.4.18) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned}
\partial^2_{t_0} u_2^{(0,1)} &= 0, && \text{in } \Omega_0, \\
u_2^{(0,1)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_2^{(1,1)}, && \text{on } S_+, \\
u_2^{(0,1)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_2^{(2,1)}, && \text{on } S_-.
\end{aligned} \tag{2.4.38}$$

Hence, the representation of $u_2^{(0,1)}(x_1, t_0)$ can be written in the form

$$u_2^{(0,1)}(x_1, t_0) = A_T^{(1)}(x_1) + \frac{t_0}{h_0} D_T^{(1)}(x_1), \tag{2.4.39}$$

where $A_T^{(1)}(x_1)$ is the average of the functions $u_2^{(1,1)}(x_1)$ and $u_2^{(2,1)}(x_1)$ evaluated at the interfaces S_+ and S_-

$$A_T^{(1)}(x_1) = \frac{u_2^{+(1,1)}(x_1) + u_2^{-(2,1)}(x_1)}{2},$$

whereas $D_T^{(1)}(x_1)$ is the displacement jump evaluated at the interfaces S_+ and S_-

$$D_T^{(1)}(x_1) = u_2^{+(1,1)}(x_1) - u_2^{-(2,1)}(x_1). \tag{2.4.40}$$

Here the subindex T denotes that the functions are related to the *transverse component*.

Proceeding in the same way with $u_1^{(1,1)}$ it is seen that Equations (2.4.13) and (2.4.17) and (2.4.19) give the following Neumann BVP on the cross-section for the upper layer,

$$\begin{aligned} \partial_{t_1}^2 u_1^{(1,1)} &= 0, && \text{in } \Omega_1, \\ \partial_{t_1} u_1^{(1,1)} &= -\partial_1 u_2^{(0)}, && \text{on } \Gamma_+, \\ \partial_{t_1} u_1^{(1,1)} &= -\partial_1 u_2^{(0)} + \frac{\mu}{\mu_1}\partial_{t_0} u_1^{(0,1)}, && \text{on } S_+. \end{aligned} \tag{2.4.41}$$

The solvability condition for the BVP (2.4.41) is given by

$$\partial_{t_0} u_1^{+(0,1)} = 0. \tag{2.4.42}$$

Proceeding in the same way with $u_1^{(2,1)}$ it is seen that the following Neumann BVP holds for the lower layer,

$$\begin{aligned} \partial_{t_2}^2 u_1^{(2,1)} &= 0, && \text{in } \Omega_2, \\ \partial_{t_2} u_1^{(2,1)} &= -\partial_1 u_2^{(0)}, && \text{on } \Gamma_-, \\ \partial_{t_2} u_1^{(2,1)} &= -\partial_1 u_2^{(0)} + \frac{\mu}{\mu_2}\partial_{t_0} u_1^{(0,1)}, && \text{on } S_-. \end{aligned} \tag{2.4.43}$$

The solvability condition for the BVP (2.4.43) is given by

$$\partial_{t_0} u_1^{-(0,1)} = 0. \tag{2.4.44}$$

Thus, the longitudinal displacements can be represented as

$$u_1^{(i,1)} = -t_i \partial_1 u_2^{(0)} + v^{(i)}(x_1), \;\; i = 1, 2, \tag{2.4.45}$$

where the functions $v^{(i)}$ are unknown and for $u_1^{(0,1)}(x_1, t_0)$ the following Dirichlet BVP on the cross-section holds

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,1)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,1)} &= u_1^{(1,1)}, && \text{on } S_+, \\ u_1^{(0,1)} &= u_1^{(2,1)}, && \text{on } S_-. \end{aligned} \tag{2.4.46}$$

Thus one finds that

$$u_1^{(0,1)}(x_1, t_0) = A_L^{(1)}(x_1) + \frac{t_0}{h_0} D_L^{(1)}(x_1), \tag{2.4.47}$$

where $A^{(1)}(x_1)$ is the average of the functions $u_1^{(1,1)}(x_1)$ and $u_1^{(2,1)}(x_1)$ evaluated at the interfaces S_+ and S_-

$$A_L^{(1)}(x_1) = \frac{u_1^{+(1,1)}(x_1) + u_1^{-(2,1)}(x_1)}{2},$$

whereas $D_L^{(1)}(x_1)$ is the longitudinal displacement jump evaluated at the interfaces S_+ and S_- and is given by

$$D_L^{(1)}(x_1) = u_1^{+(1,1)}(x_1) - u_1^{-(2,1)}(x_1).$$

Here the subindex L denotes that the functions are related to the *longitudinal component.*

Using the solvability conditions (2.4.42) and (2.4.44), the representation for $u_1^{(0,1)}(x_1, t_0)$ in Equation (2.4.47) establishes that there is no longitudinal displacement jump across the thin and soft interface in which case we write

$$u_1^{+(1,1)}(x_1) = u_1^{-(2,1)}(x_1). \tag{2.4.48}$$

Using the last Relationship one can write

$$h\partial_1 u_2^{(0)}(x_1) = v^{(2)} - v^{(1)}. \tag{2.4.49}$$

Step 3 Writing (2.4.13) for $k = 2$ and with boundary conditions (2.4.17)–(2.4.22) it is seen that the following Neumann BVP on the cross-section can be established for the upper layer

$$\begin{aligned}
\partial_{t_1}^2 u_1^{(1,2)} &= -\frac{4\mu_1 + 3\lambda_1}{\lambda_1 + 2\mu_1}\partial_1^2 u_1^{(0)}, && \text{in } \Omega_1,\\
\partial_{t_1} u_1^{(1,2)} &= \frac{t_1\lambda_1}{\lambda_1 + 2\mu_1}\partial_1^2 u_1^{(0)} - \partial_1 g^{(1)}, && \text{on } \Gamma_+,\\
\partial_{t_1} u_1^{(1,2)} &= \frac{t_1\lambda_1}{\lambda_1 + 2\mu_1}\partial_1^2 u_1^{(0)} - \partial_1 g^{(1)} && \\
&\quad + \frac{p_1^{(1)}}{\mu_1} + \frac{\mu}{\mu_1}\left(\partial_{t_0} u_1^{(0,2)} + \partial_1 u_2^{(0)}\right), && \text{on } S_+.
\end{aligned} \tag{2.4.50}$$

The solvability condition for the Neumann BVP (2.4.50) is given by

$$4\frac{h_1\mu_1(\mu_1 + \lambda_1)}{\lambda_1 + 2\mu_1}\partial_1^2 u_1^{(0)} = \mu\left(\partial_{t_0} u_1^{+(0,2)} + \partial_1 u_2^{(0)}\right) + p_1^{(1)}. \tag{2.4.51}$$

Similarly for the lower layer the following Neumann BVP can be established on the cross-section

$$\begin{aligned}
\partial^2_{t_2} u_1^{(2,2)} &= -\frac{4\mu_2 + 3\lambda_2}{\lambda_2 + 2\mu_2}\partial_1^2 u_1^{(0)}, && \text{in } \Omega_2,\\
\partial_{t_2} u_1^{(2,2)} &= \frac{t_2\lambda_2}{\lambda_2 + 2\mu_2}\partial_1^2 u_1^{(0)} - \partial_1 g^{(2)}, && \text{on } \Gamma_-,\\
\partial_{t_2} u_1^{(2,2)} &= \frac{t_2\lambda_2}{\lambda_2 + 2\mu_2}\partial_1^2 u_1^{(0)} - \partial_1 g^{(2)} \\
&\quad \frac{p_1^{(2)}}{\mu_2} + \frac{\mu}{\mu_2}\left(\partial_{t_0} u_1^{(0,2)} + \partial_1 u_2^{(0)}\right), && \text{on } S_-.
\end{aligned} \tag{2.4.52}$$

The solvability condition for the Neumann BVPs (2.4.52) is given by

$$4\frac{h_2\mu_2(\mu_2 + \lambda_2)}{\lambda_2 + 2\mu_2}\partial_1^2 u_1^{(0)} = -\mu\partial_{t_0} u_1^{+(0,2)} + \partial_1 u_2^{(0)} - p_1^{(2)}. \tag{2.4.53}$$

Using the solvability conditions (2.4.51) and (2.4.53) the solution for $u_1^{(i,2)}, i = 1, 2$ gives

$$u_1^{(1,2)} = -\frac{t_1^2(4\mu_1 + 3\lambda_1)}{2(\lambda_1 + 2\mu_1)}\partial_1^2 u_1^{(0)} + t_1\left(\frac{2h_1(\mu_1 + \lambda_1)}{\lambda_1 + 2\mu_1}\partial_1^2 u_1^{(0)} - \partial_1 g^{(1)}\right), \tag{2.4.54}$$

$$u_1^{(2,2)} = -\frac{t_2^2(4\mu_2 + 3\lambda_2)}{2(\lambda_2 + 2\mu_2)}\partial_1^2 u_1^{(0)} - t_2\left(\frac{2h_2(\mu_2 + \lambda_2)}{\lambda_2 + 2\mu_2}\partial_1^2 u_1^{(0)} + \partial_1 g^{(2)}\right). \tag{2.4.55}$$

For the middle layer, Equations (2.4.15), (2.4.17) and (2.4.18) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned}
\partial^2_{t_0} u_1^{(0,2)} &= 0, && \text{in } \Omega_0,\\
u_1^{(0,2)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_1^{(1,2)}, && \text{on } S_+,\\
u_1^{(0,2)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_1^{(2,2)}, && \text{on } S_-.
\end{aligned} \tag{2.4.56}$$

Therefore one finds that

$$u_1^{(0,2)}(x_1, t_0) = A_L^{(2)}(x_1) + \frac{t_0}{h_0}D_L^{(2)}(x_1), \tag{2.4.57}$$

where $A_L^{(2)}(x_1)$ is the average of the functions $u_1^{(1,2)}(x_1)$ and $u_1^{(2,2)}(x_1)$ evaluated at the interfaces S_+ and S_-

$$A_L^{(2)}(x_1) = \frac{u_1^{+(1,2)}(x_1) + u_1^{-(2,2)}(x_1)}{2},$$

whereas $D_L^{(2)}(x_1)$ is the longitudinal displacement jump evaluated at the interfaces S_+ and S_- and is given by

$$D_L^{(2)}(x_1) = u_1^{+(1,2)}(x_1) - u_1^{-(2,2)}(x_1).$$

Adding the Relationships (2.4.51) and (2.4.53) we find the limit equation for the leading order of the longitudinal component,

$$\boxed{4\left\{\frac{h_1(\mu_1+\lambda_1)\mu_1}{\lambda_1+2\mu_1}+\frac{h_2(\mu_2+\lambda_2)\mu_2}{\lambda_2+2\mu_2}\right\}\partial_1^2 u_1^{(0)} = p_1^{(1)} - p_1^{(2)}.} \tag{2.4.58}$$

The last Equation has been found on the basis of the asymptotic analysis. This equation coincides with the equation given by Relationship (1.4.30) since the longitudinal displacement is continuous across the middle layer.

Subtracting Relationship (2.4.53) from Relationship (2.4.51) we get the representation for $D_L^{(2)}(x_1)$

$$D_L^{(2)}(x_1) = \frac{2h_0}{\mu}\left\{\frac{h_1(\mu_1+\lambda_1)\mu_1}{\lambda_1+2\mu_1}\right\}\partial_1^2 u_1^{(0)} - h_0\partial_1 u_2^{(0)}. \tag{2.4.59}$$

Proceeding with the transverse components, from steps 1 and 2, we obtain that the following Neumann BVPs on the cross-section hold for $u_2^{(i,2)}$

$$\begin{aligned}
\partial_{t_i}^2 u_2^{(i,2)} &= \frac{\lambda_i}{2\mu_i+\lambda_i}\partial_1^2 u_2^{(0)}, && \text{in } \Omega_i,\ i=1,2,\\
\partial_{t_i} u_2^{(i,2)} &= -\frac{\lambda_i}{2\mu_i+\lambda_i}\left(\frac{h_i}{2}\partial_1^2 u_2^{(0)} + \partial_1 v^{(i)}\right) \\
&\quad +(2\mu+\lambda)\partial_{t_0} u_2^{(0,2)} + \lambda\partial_1 u_1^{(0)}, && \text{on } S_+, (i=1), S_-, (i=2),\\
\partial_{t_i} u_2^{(i,2)} &= \frac{\lambda_i}{2\mu_i+\lambda_i}\left(\frac{h_i}{2}\partial_1^2 u_2^{(0)} - \partial_1 v^{(i)}\right), && \text{on } \Gamma_+, (i=1), \Gamma_-, (i=2).
\end{aligned}$$

One can see that the solvability conditions for these problems are given by

$$\begin{aligned}
\partial_{t_0} u_2^{+(0,2)} + \lambda\partial_1 u_1^{(0)} &= 0,\\
\partial_{t_0} u_2^{-(0,2)} + \lambda\partial_1 u_1^{(0)} &= 0.
\end{aligned} \tag{2.4.60}$$

Therefore the solution for $u_2^{(i,2)}, i=1,2$ gives,

$$u_2^{(i,2)} = \frac{\lambda_i}{2\mu_i+\lambda_i}\left[\frac{t_i^2}{2}\partial_1^2 u_2^{(0)} - t_i\partial_1 v^{(i)}\right],\ i=1,2. \tag{2.4.61}$$

For the middle layer, Equations (2.4.16)–(2.4.18) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_2^{(0,2)} &= 0, && \text{in } \Omega_0, \\ u_2^{(0,2)} &= u_2^{(1,2)}, && \text{on } S_+, \\ u_2^{(0,2)} &= u_2^{(2,2)}, && \text{on } S_-. \end{aligned} \tag{2.4.62}$$

Hence one finds that the representation of $u_2^{(0,2)}(x_1, t_0)$ can be written as

$$u_2^{(0,2)}(x_1, t_0) = A_T^{(2)}(x_1) + \frac{t_0}{h_0} D_T^{(2)}(x_1), \tag{2.4.63}$$

where $A_T^{(2)}(x_1)$ is the average of the functions $u_2^{(1,2)}(x_1)$ and $u_2^{(2,2)}(x_1)$ evaluated at the interfaces S_+ and S_-,

$$A_T^{(2)}(x_1) = \frac{u_2^{+(1,2)}(x_1) + u_2^{-(2,2)}(x_1)}{2},$$

whereas $D_T^{(2)}(x_1)$ is the longitudinal displacement jump evaluated at the interfaces S_+ and S_-,

$$D_T^{(2)}(x_1) = u_2^{+(1,2)}(x_1) - u_2^{-(2,2)}(x_1).$$

Using the solvability conditions (2.4.60), we notice that $u_2^{(0,2)}(x_1, t_0)$ should be only a function of x_1 in which case $D_T^{(2)}$ must vanish.

Step 4 Writing (2.4.14) for $k = 3$ and with boundary conditions (2.4.17)–(2.4.22) it is seen that the following Neumann BVP on the cross-section can be established

$$\begin{aligned} \partial_{t_1}^2 u_2^{(1,3)} &= -\frac{\mu_1}{\lambda_1 + 2\mu_1} \partial_1^2 u_2^{(1,1)} - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} \partial_{1t_1}^2 u_1^{(1,2)}, && \text{in } \Omega_1, \\ \partial_{t_1} u_2^{(1,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1} \partial_1 u_1^{(1,2)}, && \text{on } \Gamma_+, \\ \partial_{t_1} u_2^{(1,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1} \partial_1 u_1^{(1,2)} \\ &\quad + \frac{2\mu + \lambda}{2\mu_1 + \lambda_1} \left(\partial_{t_0} u_2^{(0,3)} + \lambda \partial_1 u_1^{(0,1)} \right), && \text{on } S_+. \end{aligned} \tag{2.4.64}$$

It follows that the last Neumann BVP (2.4.64) has the following solvability condition,

$$h_1 \partial_1^2 g^{(1)} = \frac{2\mu+\lambda}{\mu_1+\lambda_1}\partial_{t_0} u_2^{+(0,3)} + \frac{\lambda}{\mu_1+\lambda_1}\partial_1 u_1^{+(0,1)} - \frac{2\mu_1 h_1^2}{\lambda_1+2\mu_1}\partial_1^3 u_1^{(0)}. \quad (2.4.65)$$

Proceeding in similar way with the lower layer, we have that the following Neumann BVP on the cross-section is satisfied

$$\begin{aligned}
\partial_{t_2}^2 u_2^{(2,3)} &= -\frac{\mu_2}{\lambda_2+2\mu_2}\partial_1^2 u_2^{(2,1)} - \frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\partial_{1t_2}^2 u_1^{(2,2)}, && \text{in } \Omega_2,\\
\partial_{t_2} u_2^{(2,3)} &= -\frac{\lambda_2}{\lambda_2+2\mu_2}\partial_1 u_1^{(2,2)}, && \text{on } \Gamma_-,\\
\partial_{t_2} u_2^{(2,3)} &= -\frac{\lambda_2}{\lambda_2+2\mu_2}\partial_1 u_1^{(2,2)} \\
&\quad + \frac{2\mu+\lambda}{2\mu_2+\lambda_2}\left(\partial_{t_0} u_2^{(0,3)} + \lambda\partial_1 u_1^{(0,1)}\right), && \text{on } S_-.
\end{aligned} \quad (2.4.66)$$

It follows that the last Neumann BVP (2.4.66) has the following solvability condition,

$$h_2 \partial_1^2 g^{(2)} = -\frac{2\mu+\lambda}{\mu_2+\lambda_2}\partial_{t_0} u_2^{-(0,3)} - \frac{\lambda}{\mu_2+\lambda_2}\partial_1 u_1^{-(0,1)} + \frac{2\mu_2 h_2^2}{\lambda_2+2\mu_2}\partial_1^3 u_1^{(0)}. \quad (2.4.67)$$

Proceeding in the same way with $u_1^{(1,3)}$ it is seen that Equations (2.4.13), (2.4.17) and (2.4.19) for $k = 3$, give the following Neumann BVP on the cross-section satisfying for the upper layer,

$$\begin{aligned}
\partial_{t_1}^2 u_1^{(1,3)} &= \frac{3\lambda_1+4\mu_1}{2\mu_1+\lambda_1}\left(t_1\partial_1^3 u_2^{(0)} - \partial_1^2 v^{(1)}\right), && \text{in } \Omega_1,\\
\partial_{t_1} u_1^{(1,3)} &= -\partial_1 u_2^{(1,2)}, && \text{on } \Gamma_+,\\
\partial_{t_1} u_1^{(1,3)} &= \frac{\mu}{\mu_1}\left(\partial_{t_0} u_1^{(0,3)}\partial_1 u_2^{+(0,1)}\right) - \partial_1 u_2^{(1,2)}, && \text{on } S_+.
\end{aligned} \quad (2.4.68)$$

It follows that the Neumann BVP (2.4.68) must satisfy this solvability condition

$$\frac{4\mu_1(\lambda_1+\mu_1)h_1}{2\mu_1+\lambda_1}\partial_1^2 v^{(1)} = \mu\left(\partial_{t_0} u_1^{+(0,3)} + \partial_1 u_2^{+(0,1)}\right) \quad (2.4.69)$$

and therefore the representation for $u_1^{(1,3)}$ can be expressed by

$$\begin{aligned} u_1^{(1,3)} = & \frac{3\lambda_1 + 4\mu_1}{2\mu_1 + \lambda_1}\left[\frac{t_1^3}{6}\partial_1^3 u_2^{(0)} - \frac{t_1^2}{2}\partial_1^2 v^{(1)}\right] \\ & - t_1 \frac{2(\lambda_1 + \mu_1)}{2\mu_1 + \lambda_1}\left[\frac{h_1^2}{4}\partial_1^3 u_2^{(0)} - h_1 \partial_1^2 v^{(1)}\right]. \end{aligned} \tag{2.4.70}$$

Similarly for $u_1^{(2,3)}$ it is found that the following Neumann BVP on the cross-section holds for the lower layer,

$$\begin{aligned} \partial_{t_2}^2 u_1^{(2,3)} &= \frac{3\lambda_2 + 4\mu_2}{2\mu_2 + \lambda_2}\left(t_2 \partial_1^3 u_2^{(0)} - \partial_1^2 v^{(2)}\right), && \text{in } \Omega_2, \\ \partial_{t_2} u_1^{(2,3)} &= -\partial_1 u_2^{(2,2)}, && \text{on } \Gamma_-, \\ \partial_{t_2} u_1^{(2,3)} &= \frac{\mu}{\mu_2}\partial_{t_0} u_1^{(0,3)} - \partial_1 u_2^{(2,2)}, && \text{on } S_-, \end{aligned}$$

which has the following solvability condition,

$$\frac{4\mu_2(\lambda_2 + \mu_2)h_2}{2\mu_2 + \lambda_2}\partial_1^2 v^{(1)} = \mu\left(\partial_{t_0} u_1^{-(0,3)} + \partial_1 u_2^{-(0,1)}\right) \tag{2.4.71}$$

Moreover, the representation for $u_1^{(2,3)}$ can be expressed by

$$\begin{aligned} u_1^{(2,3)} = & \frac{3\lambda_2 + 4\mu_2}{2\mu_2 + \lambda_2}\left[\frac{t_2^3}{6}\partial_1^3 u_2^{(0)} - \frac{t_2^2}{2}\partial_1^2 v^{(2)}\right] \\ & - t_2 \frac{2(\lambda_2 + \mu_2)}{2\mu_2 + \lambda_2}\left[\frac{h_2^2}{4}\partial_1^3 u_2^{(0)} + h_2 \partial_1^2 v^{(2)}\right]. \end{aligned} \tag{2.4.72}$$

Step 5 Writing (2.4.14) for $k = 4$ and with boundary conditions (2.4.17)–(2.4.22) it can be seen that the following Neumann BVP on the cross-section can be established

$$\begin{aligned} \partial_{t_1}^2 u_2^{(1,4)} &= -\frac{1}{2\mu_1 + \lambda_1}\left((\lambda_1 + \mu_1)\partial_{1t_1}^2 u_1^{(1,3)} + \mu_1 \partial_1^2 u_2^{(1,2)}\right), && \text{in } \Omega_1, \\ \partial_{t_1} u_2^{(1,4)} &= -\frac{\lambda_1}{2\mu_1 + \lambda_1}\partial_1 u_1^{(1,3)}, && \text{on } \Gamma_+, \\ \partial_{t_1} u_2^{(1,4)} &= \frac{2\mu + \lambda}{2\mu_1 + \lambda_1}\partial_{t_0} u_2^{(0,4)} - \frac{\lambda_1}{2\mu_1 + \lambda_1}\partial_1 u_1^{(1,3)}, && \text{on } S_+. \end{aligned} \tag{2.4.73}$$

Using the Relationship (2.4.49), we conclude that the BVP (2.4.73) is solvable if and only if:

$$\boxed{\partial_1^4 u_2^{(0)} = 0}. \tag{2.4.74}$$

Also for the lower layer it can be verified that the following BVP on the cross-section holds,

$$\begin{aligned}
\partial_{t_1}^2 u_2^{(2,4)} &= -\frac{1}{2\mu_2+\lambda_2}\left((\lambda_2+\mu_2)\partial_{1t_2}^2 u_1^{(2,3)} + \mu_2\partial_1^2 u_2^{(2,2)}\right), && \text{in } \Omega_2,\\
\partial_{t_2} u_2^{(2,4)} &= -\frac{\lambda_2}{2\mu_2+\lambda_2}\partial_1 u_1^{(2,3)}, && \text{on } \Gamma_-,\\
\partial_{t_2} u_2^{(2,4)} &= \frac{2\mu+\lambda}{2\mu_2+\lambda_2}\partial_{t_0} u_2^{(0,4)} - \frac{\lambda_2}{2\mu_2+\lambda_2}\partial_1 u_1^{(2,3)}, && \text{on } S_-.
\end{aligned} \tag{2.4.75}$$

The solvability condition for the BVP (2.4.75) is given by Relationship (2.4.74).

The clamping condition posed at the ends of the composite beam gives the boundary conditions

$$\begin{aligned}
u_2^{(0)}(\pm l) &= \partial_1 u_2^{(0)}(\pm l) = 0,\\
u_1^{(0)}(\pm l) &= 0,\ i = 1,2.
\end{aligned}$$

2.4.5 Limit equations when $E_0 = \epsilon^2 E$

For this case, the analysis is almost exactly the same as for $E_0 = \epsilon E$. The only difference is that the displacement jump will have the order $O(\epsilon)$ instead of $O(\epsilon^2)$. Here, the leading order terms for the longitudinal component of the asymptotic representation (2.4.12) are again continuous across the middle layer. This turns out to be the same for the three layers. Thus this function also satisfies the ordinary differential equation (2.4.58). Here, the transverse leading order terms are continuous and satisfy the Relationship (2.4.74).

2.4.6 Limit equations when $E_0 = \epsilon^3 E$

This is the most interesting case since the longitudinal displacement jump has the order $O(1)$. Here the case of transverse and longitudinal loading is considered. We prescribe the special type of external load where the longitudinal load is greater one order of magnitude than the transverse one,

$$\mu_1\left(\frac{\partial u_2^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_2}\right)=\epsilon^2 p_1^{(1)},\ (2\mu_1+\lambda_1)\frac{\partial u_2^{(1)}}{\partial x_2}+\lambda_1\frac{\partial u_1^{(1)}}{\partial x_1}=\epsilon^3 p_2^{(1)} \text{ on } \Gamma_+ \quad (2.4.76)$$

$$\mu_2\left(\frac{\partial u_2^{(2)}}{\partial x_1}+\frac{\partial u_1^{(2)}}{\partial x_2}\right)=\epsilon^2 p_1^{(2)},\ (2\mu_2+\lambda_2)\frac{\partial u_2^{(2)}}{\partial x_2}+\lambda_2\frac{\partial u_1^{(2)}}{\partial x_1}=\epsilon^3 p_2^{(2)} \text{ on } \Gamma_- \quad (2.4.77)$$

Indeed it is easier to bend the beam by acting a normal force on it than by acting a moment on it with the help of longitudinal forces. For further details we refer to the work by Caillerie (1984). Putting series (2.4.12) into the boundary conditions (2.4.76)–(2.4.77) we will have the following relationships for the surfaces of the compound region Ω_ϵ

$$\mu_1(\partial_{t_1}u_1^{(1,k)}+\partial_1 u_2^{(1,k-1)})=\delta_{k3}p_1^{(1)}, \quad (2.4.78)$$
$$\lambda_1\partial_1 u_1^{(1,k-1)}+(\lambda_1+2\mu_1)\partial_{t_1}u_2^{(1,k)}=\delta_{k4}p_2^{(1)} \text{ on } \Gamma_+ \quad (2.4.79)$$

and

$$\mu_2(\partial_{t_2}u_1^{(2,k)}+\partial_1 u_2^{(2,k-1)})=\delta_{k3}p_1^{(2)}, \quad (2.4.80)$$
$$\lambda_2\partial_1 u_1^{(2,k-1)}+(\lambda_2+2\mu_2)\partial_{t_2}u_2^{(2,k)}=\delta_{k4}p_2^{(2)} \text{ on } \Gamma_-. \quad (2.4.81)$$

Step 1 If $m=3$ in Equations (2.4.13), (2.4.17)–(2.4.18) and (2.4.78)–(2.4.81) it is seen that for $k=0$ the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned}\partial_{t_i}^2 u_1^{(i,0)}&=0, \text{ in } \Omega_i, i=1,2,\\ \partial_{t_i}u_1^{(i,0)}&=0, \text{ on } \Gamma_+(i=1),\Gamma_-(i=2),\\ \partial_{t_i}u_1^{(i,0)}&=0, \text{ on } S_+(i=1),S_-(i=2).\end{aligned} \quad (2.4.82)$$

The solvability conditions for the BVPs (2.4.82) are identically satisfied. Using the boundary conditions, the first terms of the approximation (2.4.12), for both the upper and lower layers, are found to be functions of x_1 only. We will assume that the leading order term of the transverse component is one order of magnitude greater that the longitudinal. Hence, it will be assumed that these functions are zero

$$u_1^{(i,0)}\equiv 0, i=1,2. \quad (2.4.83)$$

This assumption arises from the kind of load that is applied on Γ_+ and Γ_-. For the middle layer, it can be seen that the Dirichlet BVP (2.4.26) is satisfied on the cross-section. Therefore, using Equations (2.4.83), it follows that $u_1^{(0,0)}$ is zero.

$$u_1^{(0,0)} = 0.$$

When $k = 0$, Equations (2.4.14),(2.4.17)–(2.4.18) and (2.4.78)–(2.4.81) give the BVPs on the cross-section for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_2^{(i,0)} &= 0, \quad \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_2^{(i,0)} &= 0, \quad \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_2^{(i,0)} &= 0, \quad \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{2.4.84}$$

Hence the solvability conditions for the BVPs (2.4.84) are identically satisfied. With boundary conditions, the first transverse terms of the approximation (2.4.12), for both the upper and lower layers, are found to be functions of x_1 only,

$$u_2^{(i,0)} = u_2^{(i,0)}(x_1).$$

For the middle layer, it can be seen that the Dirichlet BVP (2.4.31) holds on the cross-section. Thus the representation of $u_2^{(0,0)}(x_1, t_0)$ is given by

$$u_2^{(0,0)} = A_T^{(0)} + \frac{t_0}{h_0} D_T^{(0)},$$

where $A_T^{(0)}$ is the average of the functions $u_2^{(1,0)}$ and $u_2^{(2,0)}$ evaluated at the interfaces S_+ and S_-

$$A_T^{(0)} = \frac{u_2^{+(1,0)} + u_2^{-(2,0)}}{2},$$

whereas $D_T^{(0)}$ is referred to as the displacement jump evaluated at the interfaces S_+ and S_-

$$D_T^{(0)} = u_2^{+(1,0)} - u_2^{-(2,0)}. \tag{2.4.85}$$

Step 2 For k=1 Equations (2.4.14) and (2.4.17)–(2.4.22) satisfy the following Neumann BVPs on the cross-section,

$$\begin{aligned}
\partial^2_{t_i} u_2^{(i,1)} &= 0, \text{ in } \Omega_i, i = 1,2,\\
\partial_{t_i} u_2^{(i,1)} &= 0, \text{ on } \Gamma_+(i=1), \Gamma_-(i=2),\\
\partial_{t_i} u_2^{(i,1)} &= 0, \text{ on } S_+(i=1), S_-(i=2).
\end{aligned} \tag{2.4.86}$$

The solvability condition for the Neumann BVP (2.4.86) is identically satisfied and the solution for $u_2^{(i,1)}, i = 1,2$ is given by

$$u_2^{(i,1)} = u_2^{(i,1)}(x_1).$$

For the middle layer, it can be seen that the Dirichlet BVP (2.4.38) holds on the cross-section. Hence, the representation of $u_2^{(0,1)}(x_1, t_0)$ is given by Relationship (2.4.39).

Proceeding in the same way with $u_1^{(1,1)}$ it can be verified that Equations (2.4.13), (2.4.17) and (2.4.19) give the following Neumann BVP on the cross-section for the upper layer,

$$\begin{aligned}
\partial^2_{t_1} u_1^{(1,1)} &= 0, && \text{in } \Omega_1,\\
\partial_{t_1} u_1^{(1,1)} &= -\partial_1 u_2^{(1,0)}, && \text{on } \Gamma_+,\\
\partial_{t_1} u_1^{(1,1)} &= -\partial_1 u_2^{(1,0)}, && \text{on } S_+.
\end{aligned} \tag{2.4.87}$$

The solvability condition for the BVP (2.4.87) is identically satisfied.

Proceeding in the same way with $u_1^{(2,1)}$ it can be seen that the following Neumann BVP on the cross-section holds for the lower layer,

$$\begin{aligned}
\partial^2_{t_2} u_1^{(2,1)} &= 0, && \text{in } \Omega_2,\\
\partial_{t_2} u_1^{(2,1)} &= -\partial_1 u_2^{(2,0)}, && \text{on } \Gamma_-,\\
\partial_{t_2} u_1^{(2,1)} &= -\partial_1 u_2^{(2,0)}, && \text{on } S_-.
\end{aligned} \tag{2.4.88}$$

The solvability condition for the BVP (2.4.88) is also identically satisfied. Thus, the representation for the leading order term of the longitudinal displacements can be represented as

$$u_1^{(i,1)} = -t_i \partial_1 u_2^{(i,0)} + v^{(i)}(x_1) \; i = 1,2.$$

For $u_1^{(0,1)}$, it can be verified that the Dirichlet BVP (2.4.46) on the cross-section holds. Therefore one finds that the representation of $u_1^{(0,1)}(x_1, t_0)$ is given by

Relationship (2.4.47).

Step 3 Writing (2.4.13) for $k = 2$, and with boundary conditions (2.4.17)–(2.4.18) and (2.4.78)–(2.4.81) the following Neumann BVPs on the cross-section can be established,

$$\begin{aligned}
\partial_{t_i}^2 u_1^{(i,2)} &= 0, \text{ in } \Omega_i, i = 1, 2, \\
\partial_{t_i} u_1^{(i,2)} &= -\partial_1 u_2^{(i,1)}(x_1), \text{ on } \Gamma_+(i=1), \Gamma_-(i=2), \\
\partial_{t_i} u_1^{(i,2)} &= -\partial_1 u_2^{(i,1)}(x_1), \text{ on } S_+(i=1), S_-(i=2).
\end{aligned} \tag{2.4.89}$$

The solvability conditions for the Neumann BVPs (2.4.89) are identically satisfied. The solution for $u_1^{(i,2)}, i = 1, 2$ gives

$$u_1^{(i,2)} = -t_i \partial_1 u_2^{(i,1)} + g(x_1),$$

where $g(x_1)$ is a sufficiently smooth function.

For the middle layer, Equations (2.4.15), (2.4.17) and (2.4.18) give the Dirichlet BVP (2.4.56) on the cross-section. Therefore one finds that the representation of $u_1^{(0,2)}(x_1, t_0)$ is given by Relationship (2.4.57).

Proceeding with the transverse components, from steps 1 and 2, we obtain that the following Neumann BVPs on the cross-section hold for $u_2^{(i,2)}, i = 1, 2$,

$$\begin{aligned}
\partial_{t_i}^2 u_2^{(i,2)} &= \frac{\lambda_i}{2\mu_i + \lambda_i} \partial_1^2 u_2^{(i,0)}, && \text{in } \Omega_i,\ i = 1, 2, \\
\partial_{t_i} u_2^{(i,2)} &= -\frac{\lambda_i}{2\mu_i + \lambda_i} \left(-t_i \partial_1^2 u_2^{(i,0)} + \partial_1 v^{(i)} \right), && \text{on } \Gamma_+, (i = 1), \Gamma_-, (i = 2), \\
\partial_{t_i} u_2^{(i,2)} &= -\frac{\lambda_i}{2\mu_i + \lambda_i} \left(-t_i \partial_1^2 u_2^{(i,0)} + \partial_1 v^{(i)} \right) \\
&\quad + \frac{2\mu + \lambda}{2\mu_i + \lambda_i} \partial_{t_0} u_2^{(0,0)}, && \text{on } S_+, (i = 1), S_-, (i = 2).
\end{aligned}$$

The solvability conditions for these problems are given by

$$\partial_{t_0} u_2^{(0,0)} = \frac{D_T^{(0)}}{h_0} = 0 \tag{2.4.90}$$

and so, using the Equation (2.4.85) the following notation is introduced

$$u_2^{(1,0)}(x_1) = u_2^{(2,0)}(x_1) = u_2^{(0,0)}(x_1) \equiv u_2^{(0)}(x_1).$$

Taking into account the solvability condition (2.4.90), the solution for $u_2^{(i,2)}$ is given by

$$u_2^{(i,2)} = \frac{\lambda_i}{2\mu_i + \lambda_i}\left[\frac{t_i^2}{2}\partial_1^2 u_2^{(0)} - t_i \partial_1 v^{(i)}\right], \ i = 1, 2.$$

For the middle layer, Equations (2.4.16)–(2.4.18) satisfy the Dirichlet BVP (2.4.62) on the cross-section. Thus, one finds that the representation of $u_2^{(0,2)}(x_1, t_0)$ is given by Relationship (2.4.63).

Step 4 Writing (2.4.14) for $k = 3$ and with boundary conditions (2.4.17)–(2.4.22) it is seen that the following Neumann BVP on the cross-section can be established,

$$\begin{aligned}
\partial_{t_1}^2 u_2^{(1,3)} &= -\frac{\mu_1}{\lambda_1 + 2\mu_1}\partial_1^2 u_2^{(1,1)}, \ \text{in } \Omega_1, \\
\partial_{t_1} u_2^{(1,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(1,2)}, \ \text{on } \Gamma_+, \\
\partial_{t_1} u_2^{(1,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(1,2)} + \frac{\lambda + 2\mu}{\lambda_1 + 2\mu_1}\partial_{t_0} u_2^{(0,1)}, \ \text{on } S_+.
\end{aligned} \tag{2.4.91}$$

The solvability condition of the last Neumann BVP (2.4.91) is given by

$$(\mu_1 + \lambda_1) h_0 h_1 \partial_1^2 u_2^{(1,1)}(x_1) = (\lambda + 2\mu) D_T^{(1)}(x_1). \tag{2.4.92}$$

Proceeding in similar way with the lower layer, we have that the following Neumann BVP on the cross-section is satisfied

$$\begin{aligned}
\partial_{t_2}^2 u_2^{(2,3)} &= -\frac{\mu_2}{\lambda_2 + 2\mu_2}\partial_1^2 u_2^{(2,1)}, \ \text{in } \Omega_2, \\
\partial_{t_2} u_2^{(2,3)} &= -\frac{\lambda_2}{\lambda_2 + 2\mu_2}\partial_1 u_1^{(2,2)}, \ \text{on } \Gamma_-, \\
\partial_{t_2} u_2^{(2,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(1,2)} + \frac{\lambda + 2\mu}{\lambda_1 + 2\mu_1}\partial_{t_0} u_2^{(0,1)}, \ \text{on } S_-.
\end{aligned} \tag{2.4.93}$$

The solvability condition of the last Neumann BVP (2.4.93) is given by

$$(\mu_2 + \lambda_2) h_0 h_2 \partial_1^2 u_2^{(2,1)}(x_1) = -(\lambda + 2\mu) D_T^{(1)}(x_1). \tag{2.4.94}$$

Proceeding in the same way with $u_1^{(1,3)}$ it is seen that Equations (2.4.13), (2.4.17) and (2.4.19) for $k = 3$, give the following Neumann BVP on the cross-section satisfying for the upper layer,

$$\begin{aligned}
\partial^2_{t_1} u_1^{(1,3)} &= \frac{3\lambda_1 + 4\mu_1}{2\mu_1 + \lambda_1}\left(t_1 \partial_1^3 u_2^{(0)} - \partial_1^2 v^{(1)}\right), && \text{in } \Omega_1,\\
\partial_{t_1} u_1^{(1,3)} &= -\frac{\lambda_1}{2\mu_1 + \mu_1}\left(\frac{h_1^2}{8}\partial_1^3 u_2^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)}\right) + \frac{p_1^{(1)}}{\mu_1}, && \text{on } \Gamma_+,\\
\partial_{t_1} u_1^{(1,3)} &= -\frac{\lambda_1}{2\mu_1 + \mu_1}\left(\frac{h_1^2}{8}\partial_1^3 u_2^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)}\right) \\
&\quad + \frac{\mu}{h_0}\left(\frac{h_1 + h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)}\right), && \text{on } S_+.
\end{aligned} \tag{2.4.95}$$

It follows that the Neumann BVP (2.4.95) must satisfy the following solvability condition

$$\boxed{\frac{4(\lambda_1 + \mu_1)}{2\mu_1 + \lambda_1} h_1 \partial_1^2 v^{(1)} = \frac{\mu}{\mu_1 h_0}\left[\frac{h_1 + h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)}\right] - \frac{p_1^{(1)}}{\mu_1}.} \tag{2.4.96}$$

Moreover, the representation for $u_1^{(1,3)}$ is given by

$$\begin{aligned}
u_1^{(1,3)} &= \frac{3\lambda_1 + 4\mu_1}{2\mu_1 + \lambda_1}\left[\frac{t_1^3}{6}\partial_1^3 u_2^{(0)} - \frac{t_1^2}{2}\partial_1^2 v^{(1)}\right] \\
&\quad - \frac{4(\lambda_1 + \mu_1)}{2\mu_1 + \lambda_1}\left[\frac{h_1^2}{8}\partial_1^3 u_2^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)}\right] t_1 + \frac{p_1^{(1)}}{\mu_1} t_1.
\end{aligned}$$

Also for $u_1^{(2,3)}$ it is found that the following Neumann BVP on the cross-section holds for the lower layer,

$$\begin{aligned}
\partial^2_{t_2} u_1^{(2,3)} &= \frac{3\lambda_2 + 4\mu_2}{2\mu_2 + \lambda_2}\left(t_2 \partial_1^3 u_2^{(0)} - \partial_1^2 v^{(2)}\right), && \text{in } \Omega_2,\\
\partial_{t_2} u_1^{(2,3)} &= -\frac{\lambda_2}{2\mu_2 + \mu_2}\left(\frac{h_2^2}{8}\partial_1^3 u_2^{(0)} - \frac{h_2}{2}\partial_1^2 v^{(2)}\right) + \frac{p_1^{(2)}}{\mu_2}, && \text{on } \Gamma_-,\\
\partial_{t_2} u_1^{(2,3)} &= -\frac{\lambda_2}{2\mu_2 + \mu_2}\left(\frac{h_2^2}{8}\partial_1^3 u_2^{(0)} - \frac{h_2}{2}\partial_1^2 v^{(2)}\right) \\
&\quad + \frac{\mu}{h_0}\left(\frac{h_1 + h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)}\right), && \text{on } S_-.
\end{aligned} \tag{2.4.97}$$

Thus, the Neumann BVP (2.4.97) must satisfy the following condition to have a solution

$$\boxed{\frac{4(\lambda_2+\mu_2)}{2\mu_2+\lambda_2}h_2\partial_1^2 v^{(2)} = -\frac{\mu}{\mu_2 h_0}\left[\frac{h_1+h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)}\right] + \frac{p_1^{(2)}}{\mu_2}.} \tag{2.4.98}$$

Therefore, the function $u_1^{(2,3)}$ is given by

$$\begin{aligned} u_1^{(2,3)} &= \frac{3\lambda_2+4\mu_2}{2\mu_2+\lambda_2}\left[\frac{t_2^3}{6}\partial_1^3 u_2^{(0)} - \frac{t_2^2}{2}\partial_1^2 v^{(2)}\right] \\ &\quad - \frac{4(\lambda_2+\mu_2)}{2\mu_2+\lambda_2}\left[\frac{h_2^2}{8}\partial_1^3 u_2^{(0)} + \frac{h_2}{2}\partial_1^2 v^{(2)}\right] t_2 + \frac{p_1^{(2)}}{\mu_2} t_2. \end{aligned}$$

For the middle layer we have that Equations (2.4.16)–(2.4.18) satisfy the following Dirichlet BVP on the cross-section

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,3)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,3)} &= u_1^{(1,3)}, && \text{on } S_+, \\ u_1^{(0,3)} &= u_1^{(2,3)}, && \text{on } S_-. \end{aligned}$$

Thus, one finds that the representation for $u_1^{(0,3)}(x_1, t_0)$ can be written as

$$u_1^{(0,3)} = A_L^{(3)} + \frac{t_0}{h_0} D_L^{(3)},$$

where $A_L^{(3)}$ is the average of the functions $u_1^{(1,3)}$ and $u_1^{(2,3)}$ evaluated at the interfaces S_+ and S_-

$$A_L^{(3)} = \frac{u_1^{+(1,3)} + u_1^{-(2,3)}}{2},$$

whereas $D_L^{(3)}$ is referred to as the longitudinal displacement jump evaluated at the interfaces S_+ and S_-

$$D_L^{(3)} = u_1^{+(1,3)} - u_1^{-(2,3)}.$$

Step 5 Writing (2.4.14) for $k = 4$ and with boundary conditions (2.4.17)–(2.4.18) and (2.4.78)–(2.4.81) the following Neumann BVP on the cross-section can be established

$$
\begin{aligned}
\partial_{t_1}^2 u_2^{(1,4)} &= \frac{2\mu_1 + 3\lambda_1}{2\mu_1 + \lambda_1}\left(t_1\partial_1^3 v^{(1)} - \frac{t_1^2}{2}\partial_1^4 u_2^{(0)}\right) \\
&\quad + \frac{4(\lambda_1 + \mu_1)^2}{(2\mu_1 + \lambda_1)^2}\left(\frac{h_1^2}{8}\partial_1^4 u_2^{(0)} - \frac{h_1}{2}\partial_1^3 v^{(1)}\right) \\
&\quad - \frac{\lambda_1 + \mu_1}{\mu_1(2\mu_1 + \lambda_1)}\partial_1 p_1^{(1)}, && \text{in } \Omega_1, \\
\partial_{t_1} u_2^{(1,4)} &= \frac{p_2^{(1)}}{2\mu_1 + \lambda_1} - \frac{\lambda_1}{2\mu_1 + \lambda_1}\partial_1 u_1^{(1,3)}, && \text{on } \Gamma_+, \\
\partial_{t_1} u_2^{(1,4)} &= \frac{2\mu + \lambda}{2\mu_1 + \lambda_1}\partial_{t_0} u_2^{(0,2)} - \frac{\lambda_1}{2\mu_1 + \lambda_1}\partial_1 u_1^{(1,3)} \\
&\quad + \frac{\lambda}{2\mu_1 + \lambda_1}\partial_1 u_1^{(0,1)}, && \text{on } S_+.
\end{aligned}
\tag{2.4.99}
$$

The boundary value problem (2.4.99) is solvable if and only if satisfies the following condition:

$$
\begin{aligned}
(2\mu + \lambda)\partial_{t_0} u_2^{+(0,2)} &= p_2^{(1)} + h_1\partial_1 p_1^{(1)} - \frac{1}{3}\frac{\mu_1(\mu_1 + \lambda_1)}{2\mu_1 + \lambda_1}h_1^3\partial_1^4 u_2^{(0)} \\
&\quad + 2\mu_1\frac{\lambda_1 + \mu_1}{2\mu_1 + \lambda_1}h_1^2\partial_1^3 v^{(1)}.
\end{aligned}
\tag{2.4.100}
$$

As for the lower layer it can be verified that the following BVP holds on the cross-section,

$$
\begin{aligned}
\partial_{t_2}^2 u_2^{(2,4)} &= \frac{2\mu_2 + 3\lambda_2}{2\mu_2 + \lambda_2}\left(t_2\partial_1^3 v^{(2)} - \frac{t_2^2}{2}\partial_1^4 u_2^{(0)}\right) \\
&\quad + \frac{4(\lambda_2 + \mu_2)^2}{(2\mu_2 + \lambda_2)^2}\left(\frac{h_2^2}{8}\partial_1^4 u_2^{(0)} - \frac{h_2}{2}\partial_1^3 v^{(2)}\right) \\
&\quad - \frac{\lambda_2 + \mu_2}{\mu_2(2\mu_2 + \lambda_2)}\partial_1 p_1^{(2)}, && \text{in } \Omega_2, \\
\partial_{t_2} u_2^{(2,4)} &= \frac{p_2^{(2)}}{2\mu_2 + \lambda_2} - \frac{\lambda_2}{2\mu_2 + \lambda_2}\partial_1 u_1^{(2,3)}, && \text{on } \Gamma_-, \\
\partial_{t_2} u_2^{(2,4)} &= \frac{2\mu + \lambda}{2\mu_2 + \lambda_2}\partial_{t_0} u_2^{(0,2)} - \frac{\lambda_2}{2\mu_2 + \lambda_2}\partial_1 u_1^{(2,3)}, && \text{on } S_-.
\end{aligned}
\tag{2.4.101}
$$

The BVP (2.4.101) is solvable if and only if:

$$(2\mu+\lambda)\partial_{t_0}u_2^{-(0,4)} = p_2^{(2)} - h_2\partial_1 p_1^{(2)} + \frac{1}{3}\frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\partial_1^4 u_2^{(0)} \\ +2\mu_2\frac{\lambda_2+\mu_2}{2\mu_2+\lambda_2}h_2^2\partial_1^3 v^{(2)}. \tag{2.4.102}$$

Combining the Equations (2.4.100) and (2.4.102) one can get

$$\boxed{\begin{gathered}\frac{1}{3}\left\{\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^3 + \frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\right\}\partial_1^4 u_2^{(0)} \\ +2\left\{\frac{(\lambda_2+\mu_2)\mu_2}{2\mu_2+\lambda_2}h_2^2\partial_1^3 v^{(2)} - \frac{(\lambda_1+\mu_1)\mu_1}{2\mu_1+\lambda_1}h_1^2\partial_1^3 v^{(1)}\right\} \\ = p_2^{(1)} - p_2^{(2)} + h_1\partial_1 p_1^{(1)} + h_2\partial_1 p_1^{(2)}\end{gathered}} \tag{2.4.103}$$

where the functions $v^{(i)}$ satisfy the second order differential Equations (2.4.96) and (2.4.98). The clamping condition posed at the ends of the composite beam gives the boundary conditions

$$u_2^{(0)}(\pm l) = \partial_1 u_2^{(0)}(\pm l) = 0, \\ v^{(i)}(\pm l) = 0,\ i = 1, 2.$$

It is seen that $\partial_1^3 v^{(1)}$ and $\partial_1^3 v^{(2)}$ are easily found and therefore it is possible to write the fourth-order operator (2.4.103) in a simpler form. If we differentiate (2.4.96) and (2.4.98) once with respect to x_1, and substitute $\partial_1^3 v^{(1)}, \partial_1^3 v^{(2)}$ into (2.4.103) then we get the following fourth-order ordinary differential equation for the transverse displacement

$$\boxed{\begin{gathered}\frac{1}{3}\left\{\frac{\mu_1(\mu_1+\lambda_1)h_1^3}{2\mu_1+\lambda_1} + \frac{\mu_2(\mu_2+\lambda_2)h_2^3}{2\mu_2+\lambda_2}\right\}\partial_1^4 u_2^{(0)} \\ -\frac{\mu h^2}{4h_0}\partial_1^2 u_2^{(0)} - \frac{\mu h}{2h_0}\partial_1(v^{(1)} - v^{(2)}) \\ = p_2^{(1)} - p_2^{(2)} + \frac{h_1}{2}\partial_1 p_1^{(1)} + \frac{h_2}{2}\partial_1 p_1^{(2)},\end{gathered}}$$

where $h = h_1 + h_2$. The quantity $\partial_1(v^{(1)} - v^{(2)})$ plays a crucial role in this case because it gives the unavoidable coupling effect between the transverse and longitudinal components. This is the result of the middle layer which is soft and thin. This model is new and easily implemented in a computer.

2.4.7 Summary and examples

In this Section different values of the normalised Young's modulus of the middle layer have been considered. This choice of the normalisation determines the nature of the limit equations.

First, for the case when $E_0 = \epsilon E$ (see Section 2.4.4), the limit equations (2.4.58) and (2.4.74) for the longitudinal and transverse leading components have been derived. For this case, an illustrative example of a layered structure with an imperfect interface and the property $E_0 = \epsilon E$ is given as follows. The outer layers can be made of aluminium and brass. The adhesive could be taken to be polyvinyl formal. The elastic moduli of these materials are listed in Table 2.1 where we also notice that in fact the normalised Young's modulus has the same order of magnitude as those for the adherends. The value of ϵ is fixed to be 0.1.

Second, for the case when $E_0 = \epsilon^2 E$ (see Section 2.4.5), the limit equations have been derived and they are the same that the ones for $E_0 = \epsilon E$. Here, an example of a layered structure with an imperfect interface of Young's modulus $E_0 = \epsilon^2 E$ could be where the outer layers are made of aluminium and brass; the adhesive is made of FM1000. This adhesive gives an imperfect interface softer than the one used in the previous case.

Finally, for the case when $E_0 = \epsilon^3 E$ (see Section 2.4.6), the limit equations (2.4.96) and (2.4.98) for the longitudinal leading components $v^{(1)}(x_1)$ and $v^{(2)}(x_1)$ have been derived. These equations include the leading transverse component $u_2^{(0)}$ and the quantity $v^{(1)}(x_1) - v^{(2)}(x_1)$. For the transverse leading component, the limit equation (2.4.103) has been derived. The quantity $\partial_1(v^{(1)} - v^{(2)})$ gives the unavoidable coupling effect between the transverse and longitudinal components. This is the result of the middle layer which is soft and thin. It should be remarked that the main result of the model of this Chapter is the derivation of the limit equations for the leading term components across the imperfect interface. These limit equations allow one to describe the stress and displacement components across the imperfect interface. An example of a layered structure with an imperfect interface of Young's modulus $E_0 = \epsilon^3 E$ could be where the outer layers are made of aluminium and CFRP (Carbon Fibre Reinforced Laminates); the adhesive is made of Scotweld AF-6. This adhesive gives an imperfect interface softer than the ones used in the previous cases.

Material	*Type of material*	*Young's modulus (GPA)*	*Normalised Young's modulus (GPA)*	*Poisson ratio*
Aluminium	Metal	70	—	0.30
Brass	Metal	100	—	0.25
CFRP	Carbon Fibre	135	—	0.30
Polyvinyl formal	Adhesive	2.9	29	0.17
FM1000	Adhesive	1.24	124	0.30
Scotweld AF-6	Adhesive	0.07	70	0.49

Table 2.1: Typical elastic moduli of some adherends and adhesives.

Chapter 3

Asymptotic model of a sandwich plate under general state of stress

In this chapter an asymptotic model is proposed for analysis of anisotropic, linearly elastic adhesive joints. Two layers of orthotropic material are connected by a thin and soft orthotropic adhesive: essentially the layer of adhesive can be described as a surface of discontinuity for the longitudinal displacement. This is a generalisation of Section 2.4.6 in Chapter 2. The approach (see Chapter 2) is based on the asymptotic expansion of the model displacement field. This method enables us to derive differential equations that contain a description of the displacement jump across the adhesive in terms of the leading term of the asymptotic expansion. This leading term agrees with the *engineering approach* (see Section 3.4). Derivation of limit equations for elliptic problems in thin domains using an asymptotic method was given by Leora *et al.* (1986). In particular, the limiting problem of the generalised plane stress state for the in-plane displacements and the equation for the deflection for the transverse displacement are recovered using the above technique. Also, this method has been used in the work by Nazarov (1983b) and Zorin and Romashev (1988). Other relevant works to the model here studied were made by Caillerie (1984) on thin elastic and periodic plates and Ciarlet and Destuynder (1979) on a two-dimensional linear plate theory. Also, papers by Kaprielian *et al.* (1998) and Rogers and Spencer (1989) give a general and exhaustive overview of the stretching and bending solutions for an inhomogeneous elastic plate with perfect bonding. As for the theory of plates, we refer to the monograph by Ciarlet (1997).

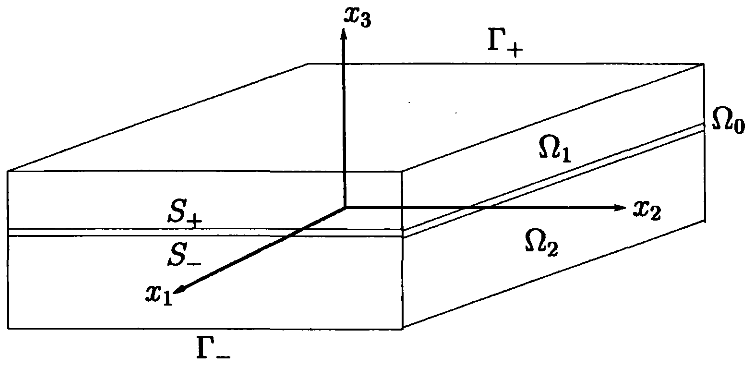

Figure 3.1: The three-dimensional thin plate Ω_ϵ.

3.1 3D layered structure with orthotropic layers

Consider a thin rectangular plate Ω_ϵ, which consists of three layers as shown in Figure 3.1:

$$\begin{aligned}
\Omega_0 = &\ \{\boldsymbol{x} \in R^3 : |x_1| < l_1,\ |x_2| < l_2\,,\ -\epsilon(h/2 - h_2) < x_3 < -\epsilon(h/2 - h_2 - \epsilon h_0)\},\\
\Omega_1 = &\ \{\boldsymbol{x} \in R^3 : |x_1| < l_1,\ |x_2| < l_2\,,\ \epsilon(h/2 - h_1) + \epsilon^2 h_0 < x_3 < \epsilon h/2 + \epsilon^2 h_0\},\\
\Omega_2 = &\ \{\boldsymbol{x} \in R^3 : |x_1| < l_1,\ |x_2| < l_2\,,\ -\epsilon h/2 < x_3 < -\epsilon h/2 + \epsilon h_2\},
\end{aligned}$$

where the quantities l_1, l_2 and h_i , $i = 0, 1, 2$, have the same order of magnitude, ϵ is a small non-dimensional positive parameter and $\boldsymbol{x} = (x_1, x_2, x_3)$. We also use the notation

$$h = h_1 + h_2.$$

The upper and lower layers have the thickness ϵh_1 and ϵh_2 respectively, and the middle layer is thinner: its thickness is equal to $\epsilon^2 h_0$. The interface boundary includes two parts, S_+ and S_-, specified by

$$\begin{aligned}
S_+ &= \{\boldsymbol{x} : |x_1| < l_1\,,\ |x_2| < l_2\,,\ x_3 = -\epsilon(h/2 - h_2) + \epsilon^2 h_0\},\\
S_- &= \{\boldsymbol{x} : |x_1| < l_1\,, |x_2| < l_2\,,\ x_3 = -\epsilon(h/2 - h_2)\}.
\end{aligned}$$

The upper and lower surfaces of the compound region are

$$\begin{aligned}
\Gamma_+ &= \{\boldsymbol{x} : |x_1| < l_1\,, |x_2| < l_2\,,\ x_3 = \epsilon^2 h_0 + \epsilon h/2\},\\
\Gamma_- &= \{\boldsymbol{x} : |x_1| < l_1\,, |x_2| < l_2\,,\,,\ x_3 = -\epsilon h/2\}.
\end{aligned}$$

The layers Ω_1 and Ω_2 are adhesively joined by the thin layer Ω_0, which is made of an adhesive material that is much softer than the material of the other two layers. If one introduces the variables

$$\begin{aligned} t_0 &= \epsilon^{-2}(x_3 + \epsilon(h/2 - h_2) - \epsilon^2 h_0/2), \\ t_1 &= \epsilon^{-1}(x_3 - \epsilon^2 h_0 - \epsilon h_2/2), \\ t_2 &= \epsilon^{-1}(x_3 + \epsilon h_1/2), \end{aligned} \tag{3.1.1}$$

one can see that

$$t_i \in [-h_i/2, h_i/2], \;\; i = 1,2; \;\; t_0 \in [-h_0/2, h_0/2]$$

and

$$\partial_3 = \epsilon^{-2}\partial_{t_0}, \;\; \partial_3 = \epsilon^{-1}\partial_{t_i}, \;\; i = 1,2,$$

where ∂_α denotes the partial derivative with respect to x_α.

3.2 The governing equations

We consider a layered structure that involves anisotropic, linearly elastic materials. Each layer is characterised by nine independent elastic moduli, since its behaviour is orthotropic with respect to the x_1, x_2, x_3 axes. We denote the displacement at any point $\boldsymbol{x}$ by $\boldsymbol{u}^{(i)}$, with components $u_i^{(i)}$, $i = 1,2,3$ and the symmetric stress tensor by σ, with components σ_{ij}.

The anisotropic response of the material is described by the stress-strain relations which can be written as

$$\begin{aligned} \sigma_{11} &= C^i_{11}\partial_1 u_1 + C^i_{12}\partial_2 u_2 + C^i_{13}\partial_3 u_3, \\ \sigma_{22} &= C^i_{12}\partial_1 u_1 + C^i_{22}\partial_2 u_2 + C^i_{23}\partial_3 u_3, \\ \sigma_{33} &= C^i_{13}\partial_1 u_1 + C^i_{23}\partial_2 u_2 + C^i_{33}\partial_3 u_3, \\ \sigma_{23} &= C^i_{44}(\partial_3 u_2 + \partial_2 u_3), \\ \sigma_{13} &= C^i_{55}(\partial_3 u_1 + \partial_1 u_3), \\ \sigma_{12} &= C^i_{66}(\partial_2 u_1 + \partial_1 u_2), \end{aligned} \tag{3.2.1}$$

where C^i_{11},C^i_{12}, $\cdots$ are the elastic moduli for Ω_i, $i = 0,1,2$. We write for convenience,

$$
\begin{array}{lll}
C^i_{11} = C^i_1, & C^i_{12} = C^i_2, & C^i_{13} = C^i_3, \\
C^i_{22} = C^i_4, & C^i_{23} = C^i_5, & C^i_{33} = C^i_6, \\
C^i_{44} = C^i_7, & C^i_{55} = C^i_8, & C^i_{66} = C^i_9,
\end{array}
$$

for $i = 0, 1, 2$. Since we consider a middle layer which is softer than the outer layers, we can choose the elastic moduli for Ω_0 to be normalised in such a way that

$$
\begin{array}{lll}
\epsilon^m \hat{C}_1 = C^0_1, & \epsilon^m \hat{C}_2 = C^0_2, & \epsilon^m \hat{C}_3 = C^0_3, \\
\epsilon^m \hat{C}_4 = C^0_4, & \epsilon^m \hat{C}_5 = C^0_5, & \epsilon^m \hat{C}_6 = C^0_6, \\
\epsilon^m \hat{C}_7 = C^0_7, & \epsilon^m \hat{C}_8 = C^0_8, & \epsilon^m \hat{C}_9 = C^0_9,
\end{array}
$$

where $m > 1$ and to illustrate the technique we fix $m = 3$. Assuming that the body forces are equal to zero, we write the equations of equilibrium in the form

$$\sum_{j=1}^{3} \partial_j \sigma_{ij} = 0. \tag{3.2.2}$$

Substituting the relations (3.2.1) into (3.2.2) we get

$$C^i_1\partial^2_1 u_1 + C^i_9\partial^2_2 u_1 + C^i_8\partial^2_3 u_1 + (C^i_2 + C^i_9)\partial^2_{12} u_2 + (C^i_3 + C^i_8)\partial^2_{13} u_3 = 0, \tag{3.2.3}$$
$$C^i_9\partial^2_1 u_2 + C^i_4\partial^2_2 u_2 + C^i_7\partial^2_3 u_2 + (C^i_2 + C^i_9)\partial^2_{12} u_1 + (C^i_5 + C^i_7)\partial^2_5 u_3 = 0, \tag{3.2.4}$$
$$C^i_8\partial^2_1 u_3 + C^i_7\partial^2_2 u_3 + C^i_6\partial^2_3 u_3 + (C^i_8 + C^i_3)\partial^2_{13} u_1 + (C^i_5 + C^i_7)\partial^2_4 u_2 = 0, \tag{3.2.5}$$

for each domain, $\Omega_i, i = 0, 1, 2$.
For the surfaces of the compound region Ω_ϵ we prescribe tractions:

$$
\begin{aligned}
C^1_8\{\partial_3 u^{(1)}_1 + \partial_1 u^{(1)}_3\} &= \epsilon^2 p^{(1)}_1, \\
C^1_7\{\partial_3 u^{(1)}_2 + \partial_2 u^{(1)}_3\} &= \epsilon^2 p^{(1)}_2, \\
C^1_3\partial_1 u^{(1)}_1 + C^1_5\partial_2 u^{(1)}_2 + C^1_6\partial_3 u^{(1)}_3 &= \epsilon^3 p^{(1)}_3 \text{ on } \Gamma_+,
\end{aligned} \tag{3.2.6}
$$

and

$$
\begin{aligned}
C^2_8\{\partial_3 u^{(2)}_1 + \partial_1 u^{(2)}_3\} &= \epsilon^2 p^{(2)}_1, \\
C^2_7\{\partial_3 u^{(2)}_2 + \partial_2 u^{(2)}_3\} &= \epsilon^2 p^{(2)}_2, \\
C^2_3\partial_1 u^{(2)}_1 + C^2_5\partial_2 u^{(2)}_2 + C^2_6\partial_3 u^{(2)}_3 &= \epsilon^3 p^{(2)}_3 \text{ on } \Gamma_-.
\end{aligned} \tag{3.2.7}
$$

The conditions of the ideal interface contact are specified by:

$$
\begin{aligned}
C_8^1\{\partial_3 u_1^{(1)} + \partial_1 u_3^{(1)}\} &= C_8^0\{\partial_3 u_1^{(0)} + \partial_1 u_3^{(0)}\},\\
C_7^1\{\partial_3 u_2^{(1)} + \partial_2 u_3^{(1)}\} &= C_7^0\{\partial_3 u_2^{(0)} + \partial_2 u_3^{(0)}\},\\
C_3^1\partial_1 u_1^{(1)} + C_5^1\partial_2 u_2^{(1)} + C_6^1\partial_3 u_3^{(1)} &= C_3^0\partial_1 u_1^{(0)} + C_5^0\partial_2 u_2^{(0)} + C_6^0\partial_3 u_3^{(0)},\\
\boldsymbol{u}^{(1)}(\boldsymbol{x}) &= \boldsymbol{u}^{(0)}(\boldsymbol{x}),\\
\boldsymbol{x} &\in S_+,
\end{aligned}
\tag{3.2.8}
$$

and

$$
\begin{aligned}
C_8^2\{\partial_3 u_1^{(2)} + \partial_1 u_3^{(2)}\} &= C_8^0\{\partial_3 u_1^{(0)} + \partial_1 u_3^{(0)}\},\\
C_7^2\{\partial_3 u_2^{(2)} + \partial_2 u_3^{(2)}\} &= C_7^0\{\partial_3 u_2^{(0)} + \partial_2 u_3^{(0)}\},\\
C_3^2\partial_1 u_1^{(2)} + C_5^2\partial_2 u_2^{(2)} + C_6^2\partial_3 u_3^{(2)} &= C_3^0\partial_1 u_1^{(0)} + C_5^0\partial_2 u_2^{(0)} + C_6^0\partial_3 u_3^{(0)},\\
\boldsymbol{u}^{(2)}(\boldsymbol{x}) &= \boldsymbol{u}^{(0)}(\boldsymbol{x}),\\
\boldsymbol{x} &\in S_-.
\end{aligned}
\tag{3.2.9}
$$

The edges of the layered structure are assumed to be fixed:

$$
\begin{aligned}
&\boldsymbol{u}^{(i)}(x_1 = \pm l_1, x_2, x_3) = \boldsymbol{0},\\
&\boldsymbol{u}^{(i)}(x_1, x_2 = \pm l_2, x_3) = \boldsymbol{0}, \quad i = 0, 1, 2.
\end{aligned}
$$

3.3 The formal asymptotic expansion

In the limit case, when the thickness of the layered structure tends to zero (or it has the order $O(\epsilon)$), the solution of the layered structure problem with an adhesive acting like glue can be represented as a linear combination of solutions of model problems corresponding to each adherend and the solution of a model problem in the adhesive region. Using the asymptotic technique, the displacement field of each thin layer can be represented as the following series (see Bakhvalov and Panasenko (1989) and Nazarov (1983b,a) who analysed homogeneous thin structures)

$$
\boldsymbol{u}^{(i)}(\boldsymbol{x}) = \sum_{k=0}^{N} \epsilon^k \boldsymbol{u}^{(i,k)}(x_1, x_2, t_i), \; i = 0, 1, 2, \tag{3.3.1}
$$

where the variables t_i are defined by Equation (3.1.1). In the text below we consider k running up to 4. The leading term of this expansion is analysed.

Putting the leading term of this series into equations (3.2.3)–(3.2.5) and boundary conditions (3.2.6)–(3.2.9), and equating the coefficients of powers of ϵ, we obtain the recurrent system of relations on the cross-section

$$
\begin{aligned}
&C_1^i\partial_1^2 u_1^{(i,k-2)} + C_6^i\partial_2^2 u_1^{(i,k-2)} + C_5^i\partial_{t_i}^2 u_1^{(i,k)} \\
&+\{C_2^i + C_9^i\}\partial_{12}^2 u_2^{(i,k-2)} + \{C_3^i + C_8^i\}\partial_{1t_i}^2 u_3^{(i,k-1)} = 0, \\
&C_9^i\partial_1^2 u_2^{(i,k-2)} + C_4^i\partial_2^2 u_2^{(i,k-2)} + C_7^i\partial_{t_i}^2 u_2^{(i,k)} \\
&+\{C_2^i + C_9^i\}\partial_{12}^2 u_1^{(i,k-2)} + \{C_5^i + C_7^i\}\partial_{2t_i}^2 u_3^{(i,k-1)} = 0, \\
&C_8^i\partial_1^2 u_3^{(i,k-2)} + C_7^i\partial_2^2 u_3^{(i,k-2)} + C_6^i\partial_{t_i}^2 u_3^{(i,k)} \\
&+\{C_8^i + C_3^i\}\partial_{1t_i}^2 u_1^{(i,k-1)} + \{C_5^i + C_7^i\}\partial_{2t_i}^2 u_2^{(i,k-1)} = 0, \quad i = 1,2,
\end{aligned}
\tag{3.3.2}
$$

for the upper and lower layers, and

$$
\begin{aligned}
&\hat{C}_1\partial_1^2 u_1^{(0,k-4)} + \hat{C}_6\partial_2^2 u_1^{(0,k-4)} + \hat{C}_5\partial_{t_0}^2 u_1^{(0,k)} \\
&+\{\hat{C}_2 + \hat{C}_9\}\partial_{12}^2 u_2^{(0,k-4)} + \{\hat{C}_3 + \hat{C}_8\}\partial_{1t_0}^2 u_3^{(0,k-2)} = 0, \\
&\hat{C}_9\partial_1^2 u_2^{(0,k-4)} + \hat{C}_4\partial_2^2 u_2^{(0,k-4)} + \hat{C}_7\partial_{t_0}^2 u_2^{(0,k)} \\
&+\{\hat{C}_2 + \hat{C}_9\}\partial_{12}^2 u_1^{(0,k-4)} + \{\hat{C}_5 + \hat{C}_7\}\partial_{2t_0}^2 u_3^{(0,k-2)} = 0, \\
&\hat{C}_8\partial_1^2 u_3^{(0,k-4)} + \hat{C}_7\partial_2^2 u_3^{(0,k-4)} + \hat{C}_6\partial_{t_0}^2 u_3^{(0,k)} \\
&+\{\hat{C}_8 + \hat{C}_3\}\partial_{1t_0}^2 u_1^{(0,k-2)} + \{\hat{C}_5 + \hat{C}_7\}\partial_{2t_0}^2 u_2^{(0,k-2)} = 0,
\end{aligned}
\tag{3.3.3}
$$

for the middle layer. On the upper interface boundary

$$
\begin{aligned}
C_8^1\{\partial_{t_1} u_1^{(1,k)} + \partial_1 u_3^{(1,k-1)}\} &= \hat{C}_8\{\partial_{t_0} u_1^{(0,k-2)} + \partial_1 u_3^{(0,k-4)}\}, \\
C_7^1\{\partial_{t_1} u_2^{(1,k)} + \partial_2 u_3^{(1,k-1)}\} &= \hat{C}_7\{\partial_{t_0} u_2^{(0,k-2)} + \partial_2 u_3^{(0,k-4)}\}, \\
C_3^1\partial_1 u_1^{(1,k-1)} + C_6^1\partial_{t_1} u_3^{(1,k)} &+ C_5^1\partial_2 u_2^{(1,k-1)} \\
&= \hat{C}_6\partial_{t_0} u_3^{(0,k-2)} + \hat{C}_3\partial_1 u_1^{(1,k-4)} + \hat{C}_5\partial_2 u_2^{(0,k-4)}, \\
u^{(0,k)} &= u^{(1,k)} \text{ on } S_+
\end{aligned}
\tag{3.3.4}
$$

and on the lower interface boundary

$$
\begin{aligned}
C_8^2\{\partial_{t_2} u_1^{(2,k)} + \partial_1 u_3^{(2,k-1)}\} &= \hat{C}_8\{\partial_{t_0} u_1^{(0,k-2)} + \partial_1 u_3^{(0,k-4)}\}, \\
C_7^2\{\partial_{t_2} u_2^{(2,k)} + \partial_2 u_3^{(2,k-1)}\} &= \hat{C}_7\{\partial_{t_0} u_2^{(0,k-2)} + \partial_2 u_3^{(0,k-4)}\}, \\
C_3^2\partial_1 u_1^{(2,k-1)} + C_6^2\partial_{t_2} u_3^{(2,k)} &+ C_5^2\partial_2 u_2^{(2,k-1)} \\
&= \hat{C}_6\partial_{t_0} u_3^{(0,k-2)} + \hat{C}_3\partial_1 u_1^{(1,k-4)} + \hat{C}_5\partial_2 u_2^{(0,k-4)}, \\
u^{(0,k)} &= u^{(2,k)} \text{ on } S_-.
\end{aligned}
\tag{3.3.5}
$$

On the upper and lower surfaces we have the following traction conditions

$$\begin{aligned} C_8^1\{\partial_{t_1} u_1^{(1,k)} + \partial_1 u_3^{(1,k-1)}\} &= \delta_{k3} p_1^{(1)}, \\ C_7^1\{\partial_{t_1} u_2^{(1,k)} + \partial_2 u_3^{(1,k-1)}\} &= \delta_{k3} p_2^{(1)}, \\ C_3^1 \partial_1 u_1^{(1,k-1)} + C_5^1 \partial_2 u_2^{(1,k-1)} + C_6^1 \partial_{t_1} u_3^{(1,k)} &= \delta_{k4} p_3^{(1)} \text{ on } \Gamma_+ \end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned} C_8^2\{\partial_{t_2} u_1^{(2,k)} + \partial_1 u_3^{(2,k-1)}\} &= \delta_{k3} p_1^{(2)}, \\ C_7^2\{\partial_{t_2} u_2^{(2,k)} + \partial_2 u_3^{(2,k-1)}\} &= \delta_{k3} p_2^{(2)}, \\ C_3^2 \partial_1 u_1^{(2,k-1)} + C_5^2 \partial_2 u_2^{(2,k-1)} + C_6^2 \partial_{t_2} u_3^{(2,k)} &= \delta_{k4} p_3^{(2)} \text{ on } \Gamma_-. \end{aligned} \tag{3.3.7}$$

Here we assume that all the terms with negative indices vanish.

3.3.1 The leading ansatz of the asymptotic expansion

We seek the following asymptotic approximation of the field $\boldsymbol{u}^{(i)}$

$$\boldsymbol{u}^{(i)}(\boldsymbol{x}) \sim \sum_{k=0}^{4} \epsilon^k \boldsymbol{u}^{(i,k)}(x_1, x_2, t_i), \;\; i = 0, 1, 2.$$

The structure of this asymptotic approximation is discussed by Nazarov (1983b) and Klarbring and Movchan (1998), for instance. Here we shall analyse all the boundary value problems for $\boldsymbol{u}^{(i)}$ and verify their solvability. Finally, the limit equations of the layered, highly inhomogeneous structure will be derived as solvability conditions for certain BVPs on the cross-section.

It is sufficient to consider the first five terms of the asymptotic series (3.3.1) in order to obtain a set of differential equations that constitute a well-posed system including the leading order components.

Step 1 If $k = 0$ in the first equation of relations (3.3.2), with boundary and interface conditions (3.3.4)–(3.3.7), it is seen the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_1^{(i,0)} &= 0, \;\; \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_1^{(i,0)} &= 0, \;\; \text{on } \Gamma_+(i=1), \Gamma_-(i=2), \\ \partial_{t_i} u_1^{(i,0)} &= 0, \;\; \text{on } S_+(i=1), S_-(i=2). \end{aligned} \tag{3.3.8}$$

It follows immediately that the solvability conditions for the BVPs (3.3.8) are identically satisfied. Also, with the boundary conditions, the first terms of the approximation (3.3.1), for both the upper and lower layers, are functions of x_1 only. It is assumed that these functions are zero as in the previous chapter, so

$$u_1^{(i,0)} \equiv 0, i = 1, 2. \tag{3.3.9}$$

For the middle layer, the first equation of relations (3.3.3) with the interface conditions (3.3.4) and (3.3.5) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,0)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,0)}\left(x_1, x_2, t_0 = \tfrac{h_0}{2}\right) &= u_1^{(1,0)}, && \text{on } S_+, \\ u_1^{(0,0)}\left(x_1, x_2, t_0 = -\tfrac{h_0}{2}\right) &= u_1^{(2,0)}, && \text{on } S_-. \end{aligned}$$

Therefore, using Equations (3.3.9), it follows that $u_1^{(0,0)}$ is zero,

$$u_1^{(0,0)} = 0.$$

For the second in-plane displacement function, substituting k=0 in the second equation of relations (3.3.2), with boundary and interface conditions (3.3.4)–(3.3.7), the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_2^{(i,0)} &= 0, \quad \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_2^{(i,0)} &= 0, \quad \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_2^{(i,0)} &= 0, \quad \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{3.3.10}$$

The solvability conditions for the BVPs (3.3.10) are identically satisfied. Also, with the boundary conditions, the first terms of the approximation (3.3.1), for both the upper and lower layers, are functions of x_1 only. Here it can be assumed that these functions are zero as in the previous chapter, so

$$u_2^{(i,0)} \equiv 0, i = 1, 2. \tag{3.3.11}$$

For the middle layer, the second equation of relations (3.3.3) with the interface conditions (3.3.4) and (3.3.5) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial^2_{t_0} u_2^{(0,0)} &= 0, && \text{in } \Omega_0, \\ u_2^{(0,0)}\left(x_1, x_2, t_0 = \tfrac{h_0}{2}\right) &= u_2^{(1,0)}, && \text{on } S_+, \\ u_2^{(0,0)}\left(x_1, x_2, t_0 = -\tfrac{h_0}{2}\right) &= u_2^{(2,0)}, && \text{on } S_-. \end{aligned}$$

Therefore, using Equations (3.3.11), it follows that $u_2^{(0,0)}$ is zero,

$$u_1^{(0,0)} = 0.$$

For k=0, the last equations of relations (3.3.2) with boundary and interface conditions (3.3.4)–(3.3.7) yield the following BVPs on the cross-section

$$\begin{aligned} \partial^2_{t_i} u_3^{(i,0)} &= 0, \ \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_3^{(i,0)} &= 0, \ \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_3^{(i,0)} &= 0, \ \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{3.3.12}$$

It follows that the solvability conditions for the BVPs (3.3.12) are identically satisfied. With boundary and interface conditions taken into account, we find that the transverse displacements are functions of x_1 and x_2 only

$$u_3^{(i,0)} = u_3^{(i,0)}(x_1, x_2), \quad i = 1, 2.$$

For the middle layer, Equations (3.3.3)–(3.3.5) give the following Dirichlet BVP on the cross-section

$$\begin{aligned} \partial^2_{t_0} u_3^{(0,0)} &= 0, && \text{in } \Omega_0, \\ u_3^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_3^{(1,0)}, && \text{on } S_+, \\ u_3^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_3^{(2,0)}, && \text{on } S_-. \end{aligned}$$

Hence, the representation of $u_3^{(0,0)}(x_1, x_2, t_0)$ can be written in the form

$$u_3^{(0,0)}(x_1, x_2, t_0) = A^{(0)}(x_1, x_2) + \frac{t_0}{h_0} D^{(0)}(x_1, x_2),$$

where $A^{(0)}(x_1, x_2)$ is the average of the functions $u_3^{(1,0)}(x_1, x_2)$ and $u_3^{(2,0)}(x_1, x_2)$ evaluated at the interfaces S_+ and S_-

$$A^{(0)}(x_1, x_2) = \frac{u_3^{+(1,0)}(x_1, x_2) + u_3^{-(2,0)}(x_1, x_2)}{2},$$

whereas $D^{(0)}(x_1, x_2)$ is the displacement jump evaluated at the interfaces S_+ and S_-

$$D^{(0)}(x_1, x_2) = u_3^{+(1,0)}(x_1, x_2) - u_3^{-(2,0)}(x_1, x_2).$$

Step 2 For k=1, the first equations of (3.3.2) with boundary and interface conditions (3.3.4)–(3.3.5) give the following BVPs on the cross-section

$$\begin{array}{lll} \partial_{t_i}^2 u_1^{(i,1)} &= 0, & \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_1^{(i,1)} &= -\partial_1 u_3^{(i,0)}, & \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_1^{(i,1)} &= -\partial_1 u_3^{(i,0)}, & \text{on } S_+(i = 1), S_-(i = 2). \end{array}$$

From these equations one obtains

$$u_1^{(i,1)}(x_1, x_2, t_i) = -t_i \partial_1 u_3^{(i,0)} + v^{(i)}(x_1, x_2), \ \ i = 1, 2.$$

For the middle layer the following Dirichlet BVP holds on the cross-section

$$\begin{array}{rll} \partial_{t_0}^2 u_1^{(0,1)} &= 0, & \text{in } \Omega_0, \\ u_1^{(0,1)} &= u_1^{(1,1)}, & \text{on } S_+, \\ u_1^{(0,1)} &= u_1^{(2,1)}, & \text{on } S_-. \end{array}$$

Then one finds that

$$u_1^{(0,1)}(x_1, x_2, t_0) = A^{(1)}(x_1, x_2) + \frac{t_0}{h_0} D^{(1)}(x_1, x_2).$$

The function $A^{(1)}(x_1, x_2)$ represents the average of the functions $u_1^{(1,1)}(x_1, x_2)$ and $u_1^{(2,1)}(x_1, x_2)$ evaluated at the interfaces S_+ and S_-

$$A^{(1)}(x_1, x_2) = \frac{u_1^{+(1,1)}(x_1, x_2) + u_1^{-(2,1)}(x_1, x_2)}{2},$$

whereas $D^{(1)}(x_1, x_2)$ is longitudinal displacement jump evaluated at the interfaces S_+ and S_-

$$D^{(1)}(x_1, x_2) = u_1^{+(1,1)}(x_1, x_2) - u_1^{-(2,1)}(x_1, x_2).$$

As for the second in-plane variables, the second equation of (3.3.2) with boundary and interface conditions (3.3.4)–(3.3.7) give the following BVPs on the cross-section

$$\begin{aligned}
\partial^2_{t_i} u_2^{(i,1)} &= 0, && \text{in } \Omega_i, i = 1, 2, \\
\partial_{t_i} u_2^{(i,1)} &= -\partial_2 u_3^{(i,0)}, && \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\
\partial_{t_i} u_2^{(i,1)} &= -\partial_2 u_3^{(i,0)}, && \text{on } S_+(i = 1), S_-(i = 2).
\end{aligned}$$

From these equations one obtains

$$u_2^{(i,1)}(x_1, x_2, t_i) = -t_i \partial_2 u_3^{(i,0)} + w^{(i)}(x_1, x_2), \ i = 1, 2.$$

For the middle layer the following Dirichlet BVP holds on the cross-section

$$\begin{aligned}
\partial^2_{t_0} u_2^{(0,1)} &= 0, && \text{in } \Omega_0, \\
u_2^{(0,1)} &= u_2^{(1,1)}, && \text{on } S_+, \\
u_2^{(0,1)} &= u_2^{(2,1)}, && \text{on } S_-.
\end{aligned}$$

Hence it follows that

$$u_2^{(0,1)}(x_1, x_2, t_0) = A^{(2)}(x_1, x_2) + \frac{t_0}{h_0} D^{(2)}(x_1, x_2),$$

where $A^{(2)}(x_1, x_2)$ is the average of the functions $u_2^{(1,1)}(x_1, x_2)$ and $u_2^{(2,1)}(x_1, x_2)$ evaluated at the interfaces S_+ and S_-

$$A^{(2)}(x_1, x_2) = \frac{u_2^{+(1,1)}(x_1, x_2) + u_2^{-(2,1)}(x_1, x_2)}{2},$$

whereas $D^{(2)}(x_1, x_2)$ is referred to as the longitudinal displacement jump evaluated at the interfaces S_+ and S_-

$$D^{(2)}(x_1, x_2) = u_2^{+(1,1)}(x_1, x_2) - u_2^{-(2,1)}(x_1, x_2).$$

Here the functions $v^{(i)}(x_1, x_2), w^{(i)}(x_1, x_2), i = 1, 2$ are sufficiently smooth.

As for the transverse displacements, for k=1 the third equations of (3.3.2), (3.3.4)–(3.3.7) satisfy the following Neumann BVPs on the cross-section,

$$\begin{aligned}
&\partial^2_{t_i} u_3^{(i,1)} = 0, \ \text{in } \Omega_i, i = 1, 2, \\
&\partial_{t_i} u_3^{(i,1)} = 0, \ \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\
&\partial_{t_i} u_3^{(i,1)} = 0, \ \text{on } S_+(i = 1), S_-(i = 2).
\end{aligned} \tag{3.3.13}$$

The solvability condition for the Neumann BVP (3.3.13) is identically satisfied and the solution for $u_3^{(i,1)}, i = 1,2$ is given by

$$u_3^{(i,1)} = u_3^{(i,1)}(x_1, x_2).$$

As in the previous step, a similar assumption is made,

$$u_3^{(i,1)} = 0, i = 1,2.$$

For the middle layer, the third equation of (3.3.3) with interface conditions (3.3.4)–(3.3.5) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_3^{(0,1)} &= 0, && \text{in } \Omega_0, \\ u_3^{(0,1)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_3^{(1,1)}, && \text{on } S_+, \\ u_3^{(0,1)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_3^{(2,1)}, && \text{on } S_-. \end{aligned}$$

Therefore, using Equations (3.3.14), it follows that $u_3^{(0,1)}$ is zero,

$$u_3^{(0,1)} = 0.$$

Step 3 Writing for the first equations in (3.3.2) for $k = 2$, and with boundary and interface conditions (3.3.4)–(3.3.7) it is seen that the following Neumann BVPs on the cross-section can be established

$$\begin{aligned} \partial_{t_i}^2 u_1^{(i,2)} &= 0, \text{ in } \Omega_i, i = 1,2, \\ \partial_{t_i} u_1^{(i,2)} &= 0, \text{ on } \Gamma_+(i=1), \Gamma_-(i=2), \\ \partial_{t_i} u_1^{(i,2)} &= 0, \text{ on } S_+(i=1), S_-(i=2). \end{aligned} \tag{3.3.14}$$

The solvability conditions for the Neumann BVPs (3.3.14) are identically satisfied. The solution for $u_1^{(i,2)}, i = 1,2$ gives

$$u_1^{(i,2)} = u_1^{(i,2)}(x_1, x_2).$$

Moreover, the following assumption is made, now for the longitudinal displacement,

$$u_1^{(i,2)} = 0, i = 1,2.$$

For the middle layer, the first equation in (3.3.3) with interface conditions

(3.3.4)–(3.3.5) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,2)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,2)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_1^{(1,2)}, && \text{on } S_+, \\ u_1^{(0,2)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_1^{(2,2)}, && \text{on } S_-. \end{aligned}$$

Therefore, using Equations (3.3.15), it follows that $u_1^{(0,2)}$ is zero,

$$u_1^{(0,1)} = 0.$$

Writing for the second equations in (3.3.2) for $k = 2$, and with boundary and interface conditions (3.3.4)–(3.3.7) it is seen that the following Neumann BVPs on the cross-section can be established for the in-plane components $u_2^{(i,2)}, i = 1, 2$

$$\begin{aligned} \partial_{t_i}^2 u_2^{(i,2)} &= 0, \text{ in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_2^{(i,2)} &= 0, \text{ on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_2^{(i,2)} &= 0, \text{ on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{3.3.15}$$

The solvability conditions for the Neumann BVPs (3.3.15) are identically satisfied. The solution for $u_2^{(i,2)}, i = 1, 2$ allows the representation

$$u_2^{(i,2)} = u_2^{(i,2)}(x_1, x_2).$$

We assume that for the longitudinal displacement,

$$u_2^{(i,2)} = 0, i = 1, 2. \tag{3.3.16}$$

For the middle layer, the second equation in (3.3.3) with interface conditions (3.3.4)–(3.3.5) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_2^{(0,2)} &= 0, && \text{in } \Omega_0, \\ u_2^{(0,2)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_2^{(1,2)}, && \text{on } S_+, \\ u_2^{(0,2)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_2^{(2,2)}, && \text{on } S_-. \end{aligned} \tag{3.3.17}$$

Hence, substituting Equations (3.3.16) into the BVP (3.3.17) it follows that $u_2^{(0,2)}$ is zero,

$$u_1^{(0,1)} = 0.$$

Using the results derived in steps 1 and 2, we conclude that the following Neumann BVP on the cross-section holds for the transverse displacements $u_3^{(i,2)}$

$$\partial_{t_i}^2 u_3^{(i,2)} = \frac{C_3^i}{C_6^i}\partial_1^2 u_3^{(i,0)} + \frac{C_5^i}{C_6^i}\partial_2^2 u_3^{(i,0)}, \text{ in } \Omega_i, i = 1,2 \tag{3.3.18}$$

with the boundary conditions

$$\begin{aligned}
\partial_{t_i} u_3^{(i,2)} &= -\frac{C_3^i}{C_6^i}\left(-t_i\partial_1^2 u_3^{(i,0)} + \partial_1 v^{(i)}\right) \\
&\quad -\frac{C_5^i}{C_6^i}\left(-t_i\partial_2^2 u_3^{(i,0)} + \partial_2 w^{(i)}\right) + \frac{\hat{C}_6}{C_6^i}\frac{D^{(0)}}{h_0}, \quad \text{on } S_+(i=1), S_-(i=2), \\
\partial_{t_i} u_3^{(i,2)} &= -\frac{C_3^i}{C_6^i}\left(-t_i\partial_1^2 u_3^{(i,0)} + \partial_1 v^{(i)}\right) \\
&\quad -\frac{C_5^i}{C_6^i}\left(-t_i\partial_2^2 u_3^{(i,0)} + \partial_2 w^{(i)}\right), \qquad \text{on } \Gamma_+(i=1), \Gamma_-(i=2).
\end{aligned} \tag{3.3.19}$$

The solvability condition for the BVP (3.3.18)–(3.3.19) is given by

$$D^{(0)} = 0,$$

that is

$$u_3^{(2,0)} = u_3^{(1,0)} = u_3^{(0,0)} \equiv u_3^{(0)}.$$

Moreover, the solution of the problem with $D^{(0)} = 0$ is given by

$$u_3^{(i,2)} = \frac{C_3^i}{C_6^i}\left(\frac{t_i^2}{2}\partial_1^2 u_3^{(0)} - t_i\partial_1 v^{(i)}\right) + \frac{C_5^i}{C_6^i}\left(\frac{t_i^2}{2}\partial_2^2 u_3^{(0)} - t_i\partial_2 w^{(i)}\right), \; i = 1,2.$$

For the middle layer, Equations (3.3.3)–(3.3.5) give the following Dirichlet BVP on the cross-section

$$\begin{aligned}
\partial_{t_0}^2 u_3^{(0,2)} &= 0, && \text{in } \Omega_0, \\
u_3^{(0,2)} &= u_3^{(1,2)}, && \text{on } S_+, \\
u_3^{(0,2)} &= u_3^{(2,2)}, && \text{on } S_-.
\end{aligned}$$

Therefore, one finds that the representation of $u_3^{(0,2)}(x_1, x_2, t_0)$ can be written in the following form

$$u_3^{(0,2)}(x_1, x_2, t_0) = A^{(3)}(x_1, x_2) + \frac{t_0}{h_0}D^{(3)}(x_1, x_2),$$

where $A^{(3)}(x_1,x_2)$ represents the average of the functions $u_3^{(1,2)}(x_1,x_2)$ and $u_3^{(2,2)}(x_1,x_2)$ evaluated at the interfaces S_+ and S_-

$$A^{(3)}(x_1,x_2) = \frac{u_3^{+(1,2)}(x_1,x_2) + u_3^{-(2,2)}(x_1,x_2)}{2},$$

whereas $D^{(3)}(x_1,x_2)$ is the longitudinal displacement jump evaluated at the interfaces S_+ and S_-

$$D^{(3)}(x_1,x_2) = u_3^{+(1,2)}(x_1,x_2) - u_3^{-(2,2)}(x_1,x_2).$$

Step 4 When k=3 one can show, similarly to the previous steps, that $u_3^{(i,3)} = 0, i = 1,2$, and thus the following Neumann BVP holds on the cross-section

$$\begin{aligned} \partial_{t_i}^2 u_1^{(i,3)} &= a_i\{t_i\partial_1^3 u_3^{(0)} - \partial_1^2 v^{(i)}\} + b_i t_i \partial_{122}^3 u_3^{(0)} \\ &\quad + c_i\partial_{12}^2 w^{(i)} - d_i\partial_2^2 v^{(i)}, \text{ in } \Omega_i, i = 1,2. \end{aligned} \tag{3.3.20}$$

The boundary conditions for the function $u_1^{(1,3)}$ are given by

$$\begin{aligned} \partial_{t_1} u_1^{(1,3)} &= \frac{p_1^{(1)}}{C_8^1} - \frac{C_3^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_1^3 u_3^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)}\right) \\ &\quad - \frac{C_5^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_{122}^3 u_3^{(0)} - \frac{h_1}{2}\partial_{12}^2 w^{(1)}\right) \text{ on } \Gamma_+, \\ \partial_{t_1} u_1^{(1,3)} &= \frac{\hat{C}_8}{C_8^1 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + v^{(1)} - v^{(2)}\right) - \frac{C_3^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_1^3 u_3^{(0)} + \frac{h_1}{2}\partial_1^2 v^{(1)}\right) \\ &\quad - \frac{C_5^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_{122}^3 u_3^{(0)} + \frac{h_1}{2}\partial_{12}^2 w^{(1)}\right) \text{ on } S_+. \end{aligned} \tag{3.3.21}$$

The solvability condition for the problem (3.3.20)–(3.3.21) has the form

$$\boxed{\begin{aligned} h_1\{d_1\partial_2^2 v^{(1)} - e_1\partial_1^2 v^{(1)} + f_1\partial_{12}^2 w^{(1)}\} = &\frac{\hat{C}_8}{C_8^1 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + v^{(1)} - v^{(2)}\right) \\ &- \frac{p_1^{(1)}}{C_8^1}, \end{aligned}} \tag{3.3.22}$$

and the function $u_1^{(1,3)}$ is given by

$$u_1^{(1,3)} = a_1\left(\frac{t_1^3}{6}\partial_1^3 u_3^{(0)} - \frac{t_1^2}{2}\partial_1^2 v^{(1)}\right) + \frac{t_1^3}{6} b_1 \partial_{122}^3 u_3^{(0)}$$
$$+\frac{t_1^2}{2}\left(c_1\partial_{12}^2 w^{(1)} - d_1\partial_2^2 v^{(1)}\right) + t_1\left\{\frac{p_1^{(1)}}{C_8^1} + \frac{h_1}{2} f_1\partial_{12}^2 w^{(1)}\right.$$
$$\left.+e_1\left(\frac{h_1^2}{8}\partial_1^3 u_3^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)}\right) + \frac{h_1^2}{8} g_1 \partial_{122}^3 u_3^{(0)} + \frac{h_1}{2} d_1 \partial_2^2 v^{(1)}\right\}.$$

We apply the same procedure for $i = 2$ and derive

$$\boxed{\begin{aligned} h_2\{d_2\partial_2^2 v^{(2)} - e_2\partial_1^2 v^{(2)} + f_2\partial_{12}^2 w^{(2)}\} = {}& -\frac{\hat{C}_8}{C_8^2 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + v^{(1)} - v^{(2)}\right) \\ & +\frac{p_1^{(2)}}{C_8^2} \end{aligned}} \tag{3.3.23}$$

and the function $u_1^{(2,3)}$ is given by

$$u_1^{(2,3)} = a_2\left(\frac{t_2^3}{6}\partial_1^3 u_3^{(0)} - \frac{t_2^2}{2}\partial_1^2 v^{(2)}\right) + \frac{t_2^3}{6} b_2 \partial_{122}^3 u_3^{(0)}$$
$$+\frac{t_2^2}{2}\left(c_1\partial_{12}^2 w^{(2)} - d_2\partial_2^2 v^{(2)}\right) + t_2\left\{\frac{p_1^{(2)}}{C_8^2} - \frac{h_2}{2} f_2\partial_{12}^2 w^{(2)}\right.$$
$$\left.+e_2\left(\frac{h_2^2}{8}\partial_1^3 u_3^{(0)} + \frac{h_2}{2}\partial_1^2 v^{(2)}\right) + \frac{h_2^2}{8} g_2 \partial_{122}^3 u_3^{(0)} - \frac{h_2}{2} d_2 \partial_2^2 v^{(2)}\right\},$$

where for convenience the following notations have been used

$$\begin{aligned}
a_i &= \frac{C_6^i C_1^i - C_3^i C_3^i - C_3^i C_8^i}{C_6^i C_8^i},\\
b_i &= \frac{2C_6^i C_9^i + C_2^i C_6^i - C_3^i C_5^i - C_5^i C_8^i}{C_6^i C_8^i},\\
c_i &= \frac{C_3^i C_5^i + C_5^i C_8^i - C_2^i C_6^i - C_6^i C_9^i}{C_6^i C_8^i},\\
d_i &= \frac{C_9^i}{C_8^i},\\
e_i &= \frac{C_3^i C_3^i - C_1^i C_6^i}{C_6^i C_8^i},\\
f_i &= \frac{C_2^i C_6^i + C_6^i C_9^i - C_3^i C_5^i}{C_6^i C_8^i},\\
g_i &= \frac{C_3^i C_5^i - 2C_6^i C_9^i - C_2^i C_6^i}{C_6^i C_8^i}.
\end{aligned}$$

Proceeding in a similar way, with $u_2^{(i,3)}$, $i = 1, 2$, one obtains the following BVPs on the cross-section

$$\begin{aligned}
\partial_{t_i}^2 u_2^{(i,3)} &= A_i\{t_i \partial_2^3 u_3^{(0)} - \partial_2^2 w^{(i)}\} + B_i t_i \partial_{112}^3 u_3^{(0)}\\
&\quad + C_i \partial_{12}^2 v^{(i)} - D_i \partial_1^2 w^{(i)}, \text{ in } \Omega_i, i = 1, 2.
\end{aligned} \tag{3.3.24}$$

The boundary conditions for the function $u_2^{(1,3)}$ are given by

$$\begin{aligned}
\partial_{t_1} u_2^{(1,3)} &= \frac{p_2^{(1)}}{C_7^1} - \frac{C_5^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_2^3 u_3^{(0)} - \frac{h_1}{2}\partial_2^2 w^{(1)}\right)\\
&\quad - \frac{C_3^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_{112}^3 u_3^{(0)} - \frac{h_1}{2}\partial_{12}^2 v^{(1)}\right) \text{ on } \Gamma_+,\\
\partial_{t_1} u_2^{(1,3)} &= \frac{\hat{C}_7}{C_7^1 h_0}\left(\frac{h_1 + h_2}{2}\partial_2 u_3^{(0)} + w^{(1)} - w^{(2)}\right)\\
&\quad - \frac{C_5^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_2^3 u_3^{(0)} + \frac{h_1}{2}\partial_2^2 w^{(1)}\right)\\
&\quad - \frac{C_3^1}{C_6^1}\left(\frac{h_1^2}{8}\partial_{112}^3 u_3^{(0)} + \frac{h_1}{2}\partial_{12}^2 v^{(1)}\right) \text{ on } S_+.
\end{aligned} \tag{3.3.25}$$

The solvability condition for the problem (3.3.24)–(3.3.25) has the form

$$\boxed{h_1\{D_1\partial_1^2 w^{(1)} - E_1\partial_2^2 w^{(1)} + F_1\partial_{12}^2 v^{(1)}\} = \frac{\hat{C}_7}{C_7^1 h_0}\left(\frac{h_1+h_2}{2}\partial_2 u_3^{(0)} + w^{(1)} - w^{(2)}\right) - \frac{p_2^{(1)}}{C_7^1}} \tag{3.3.26}$$

and the function $u_2^{(1,3)}$ is given by

$$\begin{aligned} u_2^{(1,3)} = {} & A_1\left(\frac{t_1^3}{6}\partial_2^3 u_3^{(0)} - \frac{t_1^2}{2}\partial_2^2 w^{(1)}\right) + \frac{t_1^3}{6}B_1\partial_{112}^3 u_3^{(0)} \\ & + \frac{t_1^2}{2}\left(C_1\partial_{12}^2 v^{(1)} - D_1\partial_1^2 w^{(1)}\right) + t_1\left\{\frac{p_2^{(1)}}{C_7^1} + \frac{h_1}{2}F_1\partial_{12}^2 v^{(1)}\right. \\ & \left. + E_1\left(\frac{h_1^2}{8}\partial_2^3 u_3^{(0)} - \frac{h_1}{2}\partial_2^2 w^{(1)}\right) + \frac{h_1^2}{8}G_1\partial_{112}^3 u_3^{(0)} + \frac{h_1}{2}D_1\partial_1^2 w^{(1)}\right\}. \end{aligned}$$

We apply the same procedure for $i = 2$ and derive

$$\boxed{h_2\{D_2\partial_1^2 w^{(2)} - E_2\partial_2^2 w^{(2)} + F_2\partial_{12}^2 v^{(2)}\} = -\frac{\hat{C}_7}{C_7^2 h_0}\left(\frac{h_1+h_2}{2}\partial_2 u_3^{(0)} + w^{(1)} - w^{(2)}\right) \frac{p_2^{(2)}}{C_7^2}} \tag{3.3.27}$$

and the function $u_2^{(2,3)}$ is given by

$$\begin{aligned} u_2^{(2,3)} = {} & A_2\left(\frac{t_2^3}{6}\partial_2^3 u_3^{(0)} - \frac{t_2^2}{2}\partial_2^2 w^{(2)}\right) + \frac{t_2^3}{6}B_2\partial_{112}^3 u_3^{(0)} \\ & + \frac{t_2^2}{2}\left(C_1\partial_{12}^2 v^{(2)} - D_2\partial_1^2 w^{(2)}\right) + t_2\left\{\frac{p_2^{(2)}}{C_7^2} - \frac{h_2}{2}F_2\partial_{12}^2 v^{(2)}\right. \\ & \left. + E_2\left(\frac{h_2^2}{8}\partial_2^3 u_3^{(0)} + \frac{h_2}{2}\partial_2^2 w^{(2)}\right) + \frac{h_2^2}{8}G_2\partial_{112}^3 u_3^{(0)} - \frac{h_2}{2}D_2\partial_1^2 w^{(2)}\right\}. \end{aligned}$$

Here, $A_i, B_i, \ldots$ are given by

$$
\begin{aligned}
A_i &= \frac{C_6^i C_1^i - C_5^i C_5^i - C_5^i C_7^i}{C_6^i C_7^i},\\
B_i &= \frac{2C_6^i C_9^i + C_2^i C_6^i - C_3^i C_5^i - C_3^i C_7^i}{C_6^i C_7^i},\\
C_i &= \frac{C_3^i C_5^i + C_3^i C_7^i - C_2^i C_6^i - C_6^i C_9^i}{C_6^i C_7^i},\\
D_i &= \frac{C_9^i}{C_7^i},\\
E_i &= \frac{C_5^i C_5^i - C_4^i C_6^i}{C_6^i C_7^i},\\
F_i &= \frac{C_2^i C_6^i + C_6^i C_9^i - C_3^i C_5^i}{C_6^i C_7^i},\\
G_i &= \frac{C_3^i C_5^i - 2C_6^i C_9^i - C_2^i C_6^i}{C_6^i C_7^i}.
\end{aligned}
$$

Equations (3.3.22)–(3.3.23), (3.3.26) and (3.3.27) define $v^{(1)}$, $v^{(2)}$, $w^{(1)}$ and $w^{(2)}$, provided $u_3^{(0)}$ is known. These limit equations can be compared to the ones in the literature for the case of orthotropic laminated plates.

Step 5 When k=4, the functions $u_3^{(i,4)}, i = 1,2$ satisfy the following relationships on the cross-section

$$
\begin{aligned}
\partial_{t_i}^2 u_3^{(i,4)} = & -\frac{1}{C_6^i}\left\{ C_8^i \partial_1^2 u_3^{(i,2)} + C_7^i \partial_2^2 u_3^{(i,2)} + \left(C_8^i + C_3^i \} \partial_{1t_i} u_1^{(i,3)} \right) \right\} \\
& -\frac{1}{C_6^i}\left(C_7^i + C_5^i \right) \partial_{2t_i} u_2^{(i,3)} \quad \text{in } \Omega_i,\ i = 1,2,
\end{aligned} \tag{3.3.28}
$$

and the following boundary conditions

$$
\begin{aligned}
\partial_{t_i} u_3^{(i,4)} &= \tfrac{1}{C_6^i}\left(p_3^{(i)} - C_3^i \partial_1 u_1^{(i,3)} - C_5^i \partial_2 u_2^{(i,3)} \right), && \text{on } \Gamma_+(i=1), \Gamma_-(i=2),\\
\partial_{t_i} u_3^{(i,4)} &= \tfrac{1}{C_6^i}\left(\hat{C}_6 \partial_{t_0} u_3^{(0)} - C_3^i \partial_1 u_1^{(i,3)} - C_5^i \partial_2 u_2^{(i,3)} \right), && \text{on } S_+(i=1), S_-(i=2).
\end{aligned} \tag{3.3.29}
$$

Using the representations of $u_1^{(i,3)}, u_2^{(i,3)}$ and $u_3^{(i,2)}$, $i = 1,2$ in terms of $u_3^{(0)}$ one can derive the solvability condition for the Neumann problems constituted by

(3.3.28)–(3.3.29) which is given by

$$\boxed{\begin{aligned} &-\frac{1}{12}\left\{(C_8^2 e_2 h_2^3 + C_8^1 e_1 h_1^3)\partial_1^4 u_3^{(0)} + (C_7^2 E_2 h_2^3 + C_7^1 E_1 h_1^3)\partial_2^4 u_3^{(0)}\right\} \\ &\qquad\qquad -(C_8^2 g_2 h_2^3 + C_8^1 g_1 h_1^3)\partial_{1122}^4 u_3^{(0)} \\ &-\tfrac{h_2^2}{2C_6^2}\left\{C_8^2 e_2 \partial_1^3 v^{(2)} + C_8^2 G_2(\partial_{112}^3 w^{(2)} + \partial_{122}^3 v^{(2)}) + C_7^2 E_2 \partial_2^3 w^{(2)}\right\} \\ &+\tfrac{h_1^2}{2C_6^1}\left\{C_8^1 e_1 \partial_1^3 v^{(1)} + C_8^1 G_2(\partial_{112}^3 w^{(1)} + \partial_{122}^3 v^{(1)}) + C_7^1 E_2 \partial_2^3 w^{(1)}\right\} \\ &= p_3^{(1)} - p_3^{(2)} + h_1(\tfrac{C_3^1}{C_8^1}\partial_1 p_1^{(1)} + \tfrac{C_5^1}{C_7^1}\partial_2 p_1^{(2)}) + h_2(\tfrac{C_3^2}{C_8^2}\partial_1 p_2^{(1)} + \tfrac{C_5^2}{C_7^2}\partial_2 p_2^{(2)}). \end{aligned}} \tag{3.3.30}$$

Thus Equations (3.3.22)–(3.3.23), (3.3.26)–(3.3.27) and (3.3.30) give a complete set of limit equations for the highly inhomogeneous layered structure. The clamping posed at the ends of the composite beam gives the additional boundary conditions

$$\begin{aligned} &u_3^{(0)}(\pm l_1, x_2) = \partial_1 u_3^{(0)}(\pm l_1, x_2) = 0, \\ &v^{(i)}(\pm l_1, x_2) = 0, \;\; w^{(i)}(\pm l_1, x_2) = 0; \end{aligned} \tag{3.3.31}$$

$$\begin{aligned} &u_3^{(0)}(x_1, \pm l_2) = \partial_1 u_3^{(0)}(x_1, \pm l_2) = 0, \\ &v^{(i)}(x_1, \pm l_2) = 0, \;\; w^{(i)}(x_1, \pm l_2) = 0; i = 1, 2. \end{aligned} \tag{3.3.32}$$

3.3.2 Comparison with other results

In this section we compare our limit equations for the leading order term of the asymptotic representation with those in the literature for laminated plates. Equations (3.3.22)–(3.3.23), (3.3.26)–(3.3.27) and (3.3.30) have some common terms with the limit equations derived by Rogers *et al.* (1995) for the leading order displacements.

If we replace the orthotropic by isotropic materials, in other words, if we impose the following relationships for the elastic moduli for each layer, $i = 0, 1, 2$,

$$\begin{aligned} &C_2^i = C_3^i = C_5^i = \lambda_i, \\ &C_1^i = C_4^i = C_6^i = \lambda_i + 2\mu_i, \\ &C_7^i = C_8^i = C_9^i = \tfrac{1}{2}(C_4^i - C_5^i) = \mu_i, \end{aligned}$$

it follows that Equations (3.3.22), (3.3.23), (3.3.26), (3.3.27) can be written as

$$\begin{aligned}
h_1\{\partial_2^2 v^{(1)} + a_1\partial_1^2 v^{(1)} + b_1\partial_{12}^2 w^{(1)}\} &= \frac{\mu}{\mu_1 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + v^{(1)} - v^{(2)}\right) - \frac{p_1^{(1)}}{\mu_1},\\
h_2\{\partial_2^2 v^{(2)} + a_2\partial_1^2 v^{(2)} + b_2\partial_{12}^2 w^{(2)}\} &= -\frac{\mu}{\mu_2 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + v^{(1)} - v^{(2)}\right) + \frac{p_1^{(2)}}{\mu_2},\\
h_1\{\partial_1^2 w^{(1)} + a_1\partial_2^2 w^{(1)} + b_1\partial_{12}^2 v^{(1)}\} &= \frac{\mu}{\mu_1 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + w^{(1)} - w^{(2)}\right) - \frac{p_2^{(1)}}{\mu_1},\\
h_2\{\partial_1^2 w^{(2)} + a_2\partial_2^2 w^{(2)} + b_2\partial_{12}^2 v^{(2)}\} &= -\frac{\mu}{\mu_2 h_0}\left(\frac{h_1+h_2}{2}\partial_1 u_3^{(0)} + w^{(1)} - w^{(2)}\right) + \frac{p_2^{(2)}}{\mu_2},
\end{aligned} \tag{3.3.33}$$

for the in-plane components. For convenience we have used the following notation

$$\begin{aligned}
a_i &= \frac{4(\lambda_i+\mu_i)}{2\mu_i+\lambda_i},\\
b_i &= \frac{3\lambda_i+2\mu_i}{2\mu_i+\lambda_i}.
\end{aligned}$$

Also Equation (3.3.27) is rewritten as follows

$$\begin{aligned}
&\frac{1}{3}\left\{\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^3 + \frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\right\}\Delta^2 u_3^{(0)}\\
&+2\left\{\frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^2(\partial_1\Delta v^{(2)} + \partial_2\Delta w^{(2)})\right.\\
&\left.-\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^2(\partial_1\Delta v^{(1)} + \partial_2\Delta w^{(1)})\right\} = p_3^{(1)} - p_3^{(2)} + h_1(\partial_1 p_1^{(1)} + \partial_2 p_1^{(2)})\\
&\qquad + h_2(\partial_1 p_2^{(1)} + \partial_2 p_2^{(2)}).
\end{aligned} \tag{3.3.34}$$

It is seen that the Equations (3.3.33) and (3.3.34) coincide with the ones for the two-dimensional case given by Equations (2.4.96), (2.4.98) and (2.4.104) in Chapter 2 if x_2 is cancelled.

Rogers *et al.* (1995) considered an anisotropic plate of inhomogeneous thickness for the elasticity problem and obtained in a different way, the classical and uncoupled equations for orthotropic plates, extensively studied in Christensen (1979); Lekhnitskii (1968) and Jones (1975). They defined the dimensionless material moduli by

$$Q_{ij} = \begin{cases} (c_{ij} - c_{i3}/c_{33})/c^*, & i,j = 1,2,6 \\ c^* c_{ij}/(c_{44}c_{55} - c_{45}^2), & i,j = 4,5, \end{cases}$$
$$Q_{i3} = c_{i3}/c_{33}, i = 1,2,6; Q_{33} = c^*/c_{33},$$

where c^* is a typical stiffness modulus. Their analysis led to the following differential equations,

$$M_{11}\partial_1^2 u_1^{(1)} + M_{16}\partial_1^2 u_2^{(1)} - (M_{11} + N_{11})\partial_1^3 u_3^{(0)} = l_1^{(0)},$$
$$M_{16}\partial_1^2 u_1^{(1)} + M_{66}\partial_1^2 u_2^{(1)} - (M_{16} + N_{16})\partial_1^3 u_3^{(0)} = l_2^{(0)},$$
$$N_{11}\partial_1^3 u_1^{(1)} + N_{16}\partial_1^3 u_2^{(1)} - (N_{11} + P_{11})\partial_1^4 u_3^{(0)} = l_3^{(0)},$$

where

$$M_{ij} = \int_{-1}^{1} Q_{ij}(x_3)dx_3,$$
$$N_{ij} = \int_{-1}^{1} x_3 Q_{ij}(x_3)dx_3,$$
$$P_{ij} = \int_{-1}^{1} x_3^2 Q_{ij}(x_3)dx_3.$$

The quantities $l_1^{(0)}$, $l_2^{(0)}$ and $l_3^{(0)}$ are defined in terms of the surface conditions and are given by

$$l_1^{(0)} = p_1^{(2)} - p_1^{(1)},$$
$$l_2^{(0)} = p_2^{(2)} - p_2^{(1)},$$
$$l_3^{(0)} = p_3^{(1)} - p_3^{(2)} - \partial_1 p_1^{(1)} - \partial_1 p_1^{(2)}.$$

These differential equations are the same as the usual classical laminate equations for a strip, generalised to include non-zero shear conditions on both surfaces. Elimination of $u_1^{(1)}$ and $u_2^{(1)}$ yields

$$E\partial_1^4 u_3^{(0)} = -l_3^{(0)} + L_1\partial_1 l_1^{(0)} + L_2\partial_1 l_2^{(0)}.$$

This limit equation may be recognised as the standard equation governing the normal displacement of the strip where the equivalent bending modulus E is defined by

$$E = P_{11} - (M_{11}N_{16}^2 - 2M_{16}N_{11}N_{16} + M_{66}N_{11}^2)/\triangle$$

with

$$\triangle = M_{11}M_{66} - M_{16}^2$$

and

$$\begin{aligned} L_1 &= (N_{11}M_{66} - N_{16}M_{16})/\triangle, \\ L_2 &= -(N_{11}M_{16} - N_{16}M_{11})/\triangle. \end{aligned}$$

Their approach gives higher accuracy than the classical theory. As for the strip formulation, they have shown that the transverse equation is the only one which determines the deflections of the strip, given that the solution is effectively independent of x_2. Similar results were derived for laminated strips and layered structures and compared to the results given in Jones (1975) and Christensen (1979).

In our limit equations for the in-plane displacements, we have one term involving the derivative of the transverse displacement with respect to x_1 when in their work (and in classical theory too) this term does not appear and instead, the third derivative of the transverse displacement with respect to x_1 appears in both in-plane limit equations. This difference is sensible since the middle layer has a crucial role in the mechanical behaviour of the structure. When the middle layer is replaced by a similar material, of course our approach would give similar limit equations to those in the literature. Also, it would allow a precise homogenised elastic property for the layered structure.

In this Chapter we have considered another problem, the deformation of a thin adhesive joint that involves two small parameters. For the three layered structure with an adhesive in the middle layer, that is a layer made of a thinner and softer material than the outers, it is virtually impossible to derive a single operator for the transverse displacement. It contains the functions related to the in-plane displacements and this coupling is very strong. Hence, it is not possible to consider the homogenised elastic property for this structure.

3.3.3 Remarks

The model derived by the asymptotic method can also be illustrated by the *strength of materials approach* (see Section 3.4). Obviously, the advantage of the asymptotic method is that it provides rigorous justification for the intuitive assumptions employed in Section 3.4.

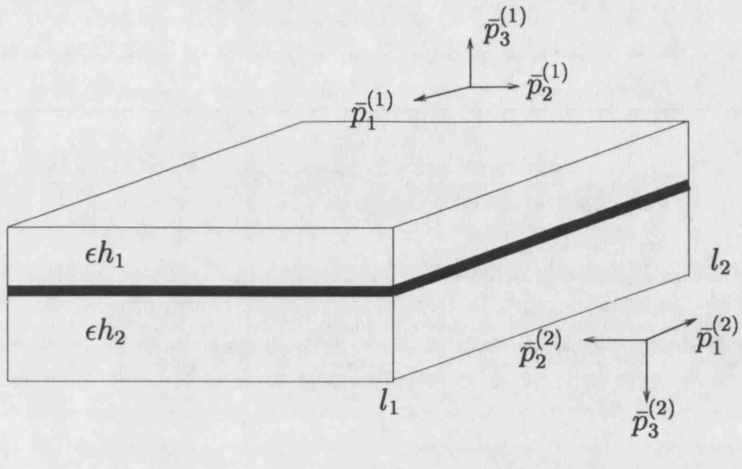

Figure 3.2: Geometry of the three-dimensional thin plate for the analysis of the strength of materials.

We note also that it has been found that the transverse displacement component is one order of magnitude larger than the longitudinal one. This is a consequence of the particular kind of external load investigated, where the difference between the orders of longitudinal and transversal load is one. Other types of external loads were dealt with by Klarbring and Movchan (1995).

3.4 Strength of materials approach

A traditional strength of materials approach is studied in this section. We consider a thin rectangular plate which consists of three layers as shown in Figure 3.2.

Considering geometry and relative softness of materials we assume that the middle layer can, due to its relative thinness and softness, be treated as a slip surface where the shear displacement may be discontinuous. It acts effectively as being rigid in the normal direction and as having the constitutive equations

$$\tau_{31} = \epsilon \frac{\hat{C}_8}{h_0} \delta_T, \tag{3.4.1}$$

$$\tau_{32} = \epsilon \frac{\hat{C}_7}{h_0} \delta_T, \tag{3.4.2}$$

when being sheared in the tangential directions. τ_{3i} are the shear stresses and δ_T is the tangential displacement jump. Here we use the same notation as in earlier sections for the elastic moduli of each layer.

Since the middle layer is treated as an infinitely thin layer one has to consider a thin rectangular plate

$$D = \ (-l_1, l_1) \times (-l_2, l_2) \times \left(-\frac{\epsilon h}{2}, \frac{\epsilon h}{2} \right) = D^1 \cup D^2,$$

$$D^1 = \ (-l_1, l_1) \times (-l_2, l_2) \times \left\{ \epsilon \left(h_2 - \frac{h}{2} \right), \frac{\epsilon h}{2} \right\},$$

$$D^2 = \ (-l_1, l_1) \times (-l_2, l_2) \times \left\{ -\frac{\epsilon h}{2}, \epsilon \left(h_2 - \frac{h}{2} \right) \right\}.$$

The displacement jump is allowed across the interface surface $x_3 = \epsilon(\frac{h}{2} - h_1)$. Here the ends of the rectangular plate D are supposed to be clamped and the load

$$\bar{p}_1^{(i)} = \epsilon^2 p_1^{(i)}, \quad \bar{p}_2^{(i)} = \epsilon^2 p_2^{(i)}, \quad \bar{p}_3^{(i)} = \epsilon^3 p_3^{(i)} \quad i = 1, 2$$

is applied on the upper and lower surfaces.

In accordance with Kirchhoff's plate theory and taking into account the assumption concerning the displacement jump, we assume the following for the displacements in D,

$$u_1 = \begin{cases} V^{(1)}(x_1, x_2) - (x_3 - \frac{\epsilon h_2}{2})\partial_1 W(x_1, x_2) \text{ in } D^1 \\ V^{(2)}(x_1, x_2) - (x_3 + \frac{\epsilon h_1}{2})\partial_1 W(x_1, x_2) \text{ in } D^2, \end{cases} \tag{3.4.3}$$

$$u_2 = \begin{cases} U^{(1)}(x_1, x_2) - (x_3 - \frac{\epsilon h_2}{2})\partial_2 W(x_1, x_2) \text{ in } D^1 \\ U^{(2)}(x_1, x_2) - (x_3 + \frac{\epsilon h_1}{2})\partial_2 W(x_1, x_2) \text{ in } D^2, \end{cases} \tag{3.4.4}$$

$$u_3 = W(x_1, x_2) \text{ in } D. \tag{3.4.5}$$

Here $V^{(i)}, U^{(i)}$ are the longitudinal displacements of the two middle axes of the layers in D^1 and D^2, and W is the transversal displacement which, due to assumptions of plate theory and due to the normal rigidity of the middle layer, is the same in the whole plate.

Using (3.4.3)–(3.4.4) we get the strain specified by

$$\varepsilon_{11} = \begin{cases} \partial_1 V^{(1)} - (x_3 - \frac{\epsilon h_2}{2})\partial_1^2 W & \epsilon(h_2 - \frac{h}{2}) < x_3 < \frac{\epsilon h}{2} \\ \partial_1 V^{(2)} - (x_3 + \frac{\epsilon h_1}{2})\partial_1^2 W & -\frac{\epsilon h}{2} < x_3 < \epsilon(h_2 - \frac{h}{2}), \end{cases} \tag{3.4.6}$$

$$\varepsilon_{22} = \begin{cases} \partial_2 U^{(1)} - (x_3 - \frac{\epsilon h_2}{2})\partial_2^2 W & \epsilon(h_2 - \frac{h}{2}) < x_3 < \frac{\epsilon h}{2} \\ \partial_2 U^{(2)} - (x_3 + \frac{\epsilon h_1}{2})\partial_1^2 W & -\frac{\epsilon h}{2} < x_3 < \epsilon(h_2 - \frac{h}{2}), \end{cases} \tag{3.4.7}$$

$$\varepsilon_{12} = \frac{1}{2} \begin{cases} \partial_2 V^{(1)} - 2(x_3 - \frac{\epsilon h_2}{2})\partial_{12}^2 W + \partial_1 U^{(1)} & \epsilon(h_2 - \frac{h}{2}) < x_3 < \frac{\epsilon h}{2} \\ \partial_2 V^{(2)} - 2(x_3 + \frac{\epsilon h_1}{2})\partial_{12}^2 W + \partial_1 U^{(2)} & -\frac{\epsilon h}{2} < x_3 < \epsilon(h_2 - \frac{h}{2}). \end{cases} \tag{3.4.8}$$

Given the kinematic assumption expressed in (3.4.3)–(3.4.5), the principle of virtual work will provide the equilibrium equations for the plate. Using the notation σ_{ij} for the stress, this principle is

$$\begin{aligned}
&\int\!\!\int\!\!\int_D (\sigma_{11}, \sigma_{22}, 2\sigma_{12}) \cdot (\delta\varepsilon_{11}, \delta\varepsilon_{22}, \delta\varepsilon_{12}) dV \\
&+ \int\!\!\int_S \tau_{31} \left[\delta u_1 \left(x_1, x_2, \epsilon \left(h_2 - \frac{h}{2} \right) + 0 \right) - \delta u_1 \left(x_1, x_2, \epsilon \left(h_2 - \frac{h}{2} \right) - 0 \right) \right] dS \\
&+ \int\!\!\int_S \tau_{32} \left[\delta u_2 \left(x_1, x_2, \epsilon \left(h_2 - \frac{h}{2} \right) + 0 \right) - \delta u_2 \left(x_1, x_2, \epsilon \left(h_2 - \frac{h}{2} \right) - 0 \right) \right] dS \\
&= \int\!\!\int_S \Bigg\{ (\bar{p}_3^{(1)} - \bar{p}_3^{(2)}) \delta W + (\bar{p}_1^{(1)}, \bar{p}_1^{(2)}) \cdot \left(\delta u_1 \left(x_1, x_2, \frac{\epsilon h}{2} \right), \delta u_1 \left(x_1, x_2, \frac{-\epsilon h}{2} \right) \right) \\
&+ (\bar{p}_2^{(1)}, \bar{p}_2^{(2)}) \cdot \left(\delta u_2 \left(x_1, x_2, \frac{\epsilon h}{2} \right), \delta u_2 \left(x_1, x_2, \frac{-\epsilon h}{2} \right) \right) \Bigg\} dS, \qquad (3.4.9)
\end{aligned}$$

for all virtual displacements $(\delta u_1, \delta u_2)$ and all virtual strains $(\delta\varepsilon_{11}, \delta\varepsilon_{22}, \delta\varepsilon_{12})$ compatible with (3.4.6)–(3.4.8) and clamping conditions at $x_1 = \pm l_1, x_2 = \pm l_2$. Here we write $S = (-l_1, l_1) \times (-l_2, l_2)$. Also, we introduce the following notations

$$\begin{aligned}
H_- &= \epsilon(h_2 - \tfrac{h}{2}), & H_+ &= \epsilon\tfrac{h}{2} \\
t &= x_3 - \tfrac{\epsilon h_2}{2}, & w &= x_3 + \tfrac{\epsilon h_1}{2}.
\end{aligned}$$

By substitution of Equations (3.4.3) and (3.4.4) we find that Equation (3.4.9) can be written as

$$\begin{aligned}
\int\!\!\int_S & \left\{ \int_{H_-}^{H_+} \left(\sigma_{11}(\partial_1 \delta V^{(1)} - t\partial_1^2 \delta W) + \sigma_{22}(\partial_2 \delta U^{(1)} - t\partial_2^2 \delta W) \right. \right. \\
& \qquad \left. + 2\sigma_{12}(\partial_2 \delta V^{(1)} - 2t\partial_{12}^2 \delta W + \partial_1 \delta U^{(1)}) \right) dx_3 \\
& + \int_{-H_+}^{H_-} \left(\sigma_{11}(\partial_1 \delta V^{(2)} - w\partial_1^2 \delta W) + \sigma_{22}(\partial_2 \delta U^{(2)} - w\partial_2^2 \delta W) \right. \\
& \qquad \left. + 2\sigma_{12}(\partial_2 \delta V^{(2)} - 2w\partial_{12}^2 \delta W + \partial_1 \delta U^{(2)}) \right) dx_3 \\
& + \tau_{31} \left(\delta V^{(1)} - \delta V^{(2)} + \epsilon \frac{h}{2} \partial_1 \delta W \right) \\
& \left. + \tau_{32} \left(\delta U^{(1)} - \delta U^{(2)} + \epsilon \frac{h}{2} \partial_2 \delta W \right) \right\} dS \\
= \int\!\!\int_S & \left\{ (\bar{p}_3^{(1)} - \bar{p}_3^{(2)}) \delta W + \bar{p}_1^{(1)} \left(\delta V_1 - \frac{\epsilon h_1}{2} \partial_1 \delta W \right) \right. \\
& - \bar{p}_1^{(2)} \left(\delta V_2 + \frac{\epsilon h_2}{2} \partial_1 \delta W \right) + \bar{p}_2^{(1)} \left(\epsilon U_1 - \frac{\epsilon h_1}{2} \partial_2 \delta W \right) \\
& \left. - \bar{p}_2^{(2)} \left(\delta U_2 + \frac{\epsilon h_2}{2} \partial_2 \delta W \right) \right\} dS. \qquad (3.4.10)
\end{aligned}$$

Now we introduce normal forces and bending moments given by

$$\begin{array}{ll}
N_1^{(1)} = \int_{H_-}^{H_+} \sigma_{11} dx_3, & N_1^{(2)} = \int_{-H_+}^{H_-} \sigma_{11} dx_3, \\
N_2^{(1)} = \int_{H_-}^{H_+} \sigma_{22} dx_3, & N_2^{(2)} = \int_{-H_+}^{H_-} \sigma_{22} dx_3, \\
N_3^{(1)} = \int_{H_-}^{H_+} \sigma_{12} dx_3, & N_3^{(2)} = \int_{-H_+}^{H_-} \sigma_{12} dx_3, \\
M_1^{(1)} = \int_{H_-}^{H_+} -t\sigma_{11} dx_3, & M_1^{(2)} = \int_{-H_+}^{H_-} -w\sigma_{11} dx_3, \\
M_2^{(1)} = \int_{H_-}^{H_+} -t\sigma_{22} dx_3, & M_2^{(2)} = \int_{-H_+}^{H_-} -w\sigma_{22} dx_3, \\
M_3^{(1)} = \int_{H_-}^{H_+} -t\sigma_{12} dx_3, & M_3^{(2)} = \int_{-H_+}^{H_-} -w\sigma_{12} dx_3.
\end{array} \qquad (3.4.11)$$

Substituting these expressions (3.4.11) into Equation (3.4.10) and integrating by parts we obtain

$$
\begin{aligned}
\int\!\!\int_S \Bigg\{ & -\delta V^{(1)}\partial_1 N_1^{(1)} + \delta W \partial_1^2 (M_1^{(1)} + M_1^{(2)}) - \delta V^{(2)}\partial_1 N_1^{(2)} - \delta U^{(1)}\partial_1 N_3^{(1)} \\
& -\delta U^{(2)}\partial_1 N_3^{(2)} - \delta U^{(1)}\partial_2 N_2^{(1)} + \delta W \partial_2^2 (M_2^{(1)} + M_2^{(2)}) - \delta U^{(2)}\partial_1 N_2^{(2)} \\
& -\delta V^{(1)}\partial_2 N_3^{(1)} - \delta V^{(2)}\partial_2 N_3^{(2)} + 2\delta W \partial_{12}^2 (M_3^{(1)} + M_3^{(2)}) \\
& +\tau_{31}\{\delta V^{(1)} - \delta V^{(2)}\} - \epsilon\frac{h}{2}\delta W\{\partial_1\tau_{31} + \partial_2\tau^{32}\} + \tau_{32}\{\delta U^{(1)} - \delta U^{(2)}\} \\
& -\delta W\{\bar{p}_3^{(1)} - \bar{p}_3^{(2)} + \epsilon\frac{h_1}{2}\partial_1\bar{p}_1^{(1)} + \epsilon\frac{h_2}{2}\partial_1\bar{p}_1^{(2)} + \epsilon\frac{h_1}{2}\partial_2\bar{p}_2^{(1)} + \epsilon\frac{h_2}{2}\partial_2\bar{p}_2^{(2)}\} \\
& -\delta V^{(1)} p_1^{(1)} + \delta V^{(2)} p_1^{(2)} - \delta U^{(1)} p_2^{(1)} + \delta U^{(2)} p_2^{(2)} \Bigg\} dS = 0. \qquad (3.4.12)
\end{aligned}
$$

Taking $\delta V^{(i)}$, $\delta U^{(i)}$, δW to be arbitrary we obtain the equilibrium equations

$$-\partial_1 N_1^{(1)} - \partial_2 N_3^{(1)} + \tau_{31} - \bar{p}_1^{(1)} = 0, \qquad (3.4.13)$$

$$-\partial_1 N_1^{(2)} - \partial_2 N_3^{(2)} - \tau_{31} + \bar{p}_1^{(2)} = 0, \qquad (3.4.14)$$

$$-\partial_1 N_3^{(1)} - \partial_2 N_2^{(1)} + \tau_{32} - \bar{p}_2^{(1)} = 0, \qquad (3.4.15)$$

$$-\partial_1 N_3^{(2)} - \partial_2 N_2^{(2)} - \tau_{32} + \bar{p}_2^{(2)} = 0, \qquad (3.4.16)$$

$$
\begin{aligned}
& \partial_1^2 (M_1^{(1)} + M_1^{(2)}) + \partial_2^2 (M_2^{(1)} + M_2^{(2)}) \\
& +2\partial_{12}^2 (M_3^{(1)} + M_3^{(2)}) - (\bar{p}_3^{(1)} - \bar{p}_3^{(2)}) \\
& -\frac{\epsilon}{2}\Bigg\{ \partial_1 \left(h_1\bar{p}_1^{(1)} + h_2\bar{p}_1^{(2)} + h\tau_{31} \right) \\
& +\partial_2 \left(h_1\bar{p}_2^{(1)} + h_2\bar{p}_2^{(2)} + h\tau_{32} \right) \Bigg\} = 0. \qquad (3.4.17)
\end{aligned}
$$

Taking into account (3.4.3)–(3.4.4) one can see that (3.4.1)–(3.4.2) can be written as

$$
\begin{aligned}
\tau_{31} &= \epsilon\frac{\hat{C}_8}{h_0}\left(u_1(x_1, x_2, H_- + 0) - u_1(x_1, x_2, H_- - 0)\right) \\
&= \epsilon\frac{\hat{C}_8}{h_0}\left(V^{(1)} - V^{(2)} + \epsilon\frac{h}{2}\partial_1 W\right), \qquad (3.4.18) \\
\tau_{32} &= \epsilon\frac{\hat{C}_7}{h_0}\left(u_2(x_1, x_2, H_- + 0) - u_2(x_1, x_2, H_- - 0)\right) \\
&= \epsilon\frac{\hat{C}_7}{h_0}\left(U^{(1)} - U^{(2)} + \epsilon\frac{h}{2}\partial_2 W\right). \qquad (3.4.19)
\end{aligned}
$$

For the stresses in D we have that

$$\sigma_{11} = \begin{cases} C_1^1\{\partial_1 V^{(1)} - t\partial_1^2 W\} + C_2^1\{\partial_2 U^{(1)} - t\partial_2^2 W\} & -\frac{\epsilon h_1}{2} < t < \frac{\epsilon h_1}{2}, \\ C_1^2\{\partial_1 V^{(2)} - w\partial_1^2 W\} + C_2^2\{\partial_2 U^{(2)} - w\partial_2^2 W\} & -\frac{\epsilon h_2}{2} < w < \frac{\epsilon h_2}{2}, \end{cases}$$

$$\sigma_{22} = \begin{cases} C_2^1\{\partial_1 V^{(1)} - t\partial_1^2 W\} + C_4^1\{\partial_2 U^{(1)} - t\partial_2^2 W\} & -\frac{\epsilon h_1}{2} < t < \frac{\epsilon h_1}{2}, \\ C_2^2\{\partial_1 V^{(2)} - w\partial_1^2 W\} + C_4^2\{\partial_2 U^{(2)} - w\partial_2^2 W\} & -\frac{\epsilon h_2}{2} < w < \frac{\epsilon h_2}{2}, \end{cases}$$

$$\sigma_{12} = \begin{cases} C_9^1\{\partial_2 V^{(1)} + \partial_1 U^{(1)} - 2t\partial_{12}^2 W\} & -\frac{\epsilon h_1}{2} < t < \frac{\epsilon h_1}{2}, \\ C_9^2\{\partial_2 V^{(2)} + \partial_1 U^{(2)} - 2w\partial_{12}^2 W\} & -\frac{\epsilon h_2}{2} < w < \frac{\epsilon h_2}{2}. \end{cases}$$

Substituting these relations into the expressions for normal forces and bending moments one obtains

$$\begin{array}{ll} N_1^{(1)} = C_1^1\epsilon h_1\partial_1 V^{(1)} + C_2^1\epsilon h_1\partial_2 U^{(1)}, & N_1^{(2)} = C_1^2\epsilon h_2\partial_1 V^{(2)} + C_2^2\epsilon h_2\partial_2 U^{(2)} \\ N_2^{(1)} = C_2^1\epsilon h_1\partial_1 V^{(1)} + C_4^1\epsilon h_1\partial_2 U^{(1)}, & N_2^{(2)} = C_2^2\epsilon h_2\partial_1 V^{(2)} + C_4^2\epsilon h_2\partial_2 U^{(2)} \\ N_3^{(1)} = C_9^1\epsilon h_1\{\partial_2 V^{(1)} + \partial_1 U^{(1)}\}, & N_3^{(2)} = C_9^2\epsilon h_2\{\partial_2 V^{(2)} + \partial_1 U^{(2)}\} \\ M_1^{(1)} = C_1^1\frac{\epsilon^3 h_1^3}{12}\partial_1^2 W + C_2^1\frac{\epsilon^3 h_1^3}{12}\partial_2^2 W, & M_1^{(2)} = C_1^2\frac{\epsilon^3 h_2^3}{12}\partial_1^2 W + C_2^2\frac{\epsilon^3 h_2^3}{12}\partial_2^2 W \\ M_2^{(1)} = C_2^1\frac{\epsilon^3 h_1^3}{12}\partial_1^2 W + C_4^1\frac{\epsilon^3 h_1^3}{12}\partial_2^2 W, & M_2^{(2)} = C_2^2\frac{\epsilon^3 h_2^3}{12}\partial_1^2 W + C_4^2\frac{\epsilon^3 h_2^3}{12}\partial_2^2 W \\ M_3^{(1)} = C_9^1\frac{\epsilon^3 h_1^3}{6}\partial_{12}^2 W, & M_3^{(2)} = C_9^2\frac{\epsilon^3 h_2^3}{6}\partial_{12}^2 W. \end{array} \tag{3.4.20}$$

Substituting (3.4.18)–(3.4.20) into (3.4.13)–(3.4.17) one obtains a system which is easily seen to be identical to (3.3.22)–(3.3.23), (3.3.26)–(3.3.27) and (3.3.30) provided $W, V^{(i)}$, and $U^{(i)}$ are identified with $u_3^{(0)}, \epsilon v^{(i)}$, and $\epsilon w^{(i)}$, respectively. Thus, the model derived by the asymptotic method is found to agree completely with the one found in this Section.

3.5 Numerical Examples

The two-dimensional case is considered here for the sake of simplicity. Both the isotropic and orthotropic cases are treated. We solve the system composed from relations (3.3.22), (3.3.23) and (3.3.30) with the corresponding boundary conditions (3.3.31). In the corresponding system all derivatives with respect to x_2 vanish and also the functions w_1, w_2 are not involved. Consider the case where the second component of the load is zero ($p_2^{(i)} = 0, i = 1, 2$). We present the test with different values of $p_1^{(1)}, p_3^{(1)}, p_1^{(2)}, p_3^{(2)}$ as shown in Table 3.1.

Isotropic Case. In this case, it is well known that the problem involves 2 elastic parameters,the so-called Lamé constants.

The elastic materials of the regions $\Omega_i, i = 0, 1, 2$ are characterised by the

Case	$p_1^{(1)}$	$p_1^{(2)}$	$p_3^{(1)}$	$p_3^{(2)}$
1	1000	0	0	0
2	0	0	1000	0
3	1000	1000	0	0
4	1000	−1000	0	0

Table 3.1: The loading cases.

Young's moduli E_i and by the values ν_i of the Poisson ratio, $i = 0, 1, 2$. We assume that $E_0 = \epsilon^3 E$, where E is comparable with E_1,E_2 and that $\nu_i \equiv \nu, i = 0, 1, 2$ is the same for the three layers (see Figures 3.3–3.8). By

$$\lambda_i, \quad \mu_i, i = 0, 1, 2$$

we denote the Lamé constants of the elastic materials

$$\lambda_i = \frac{E_i \nu_i}{(1+\nu_i)(1-2\nu_i)}, \mu_i = \frac{E_i}{2(1+\nu_i)},$$

while

$$\lambda = \frac{E\nu_0}{(1+\nu_0)(1-2\nu_0)}, \mu = \frac{E}{2(1+\nu_0)}$$

are the normalised Lamé constants related to the middle layer. We choose the following values of parameters

$$\begin{array}{ll} l_1 = 1, & \epsilon = 0.01, \\ h_1 = 2, & h_2 = 3, \\ h_0 = 1, & E_1 = 2\text{x}10^6, \\ E_2 = 3\text{x}10^6, & E_0 = 1\text{x}10^6, \\ \nu = 0.35. & \end{array}$$

Orthotropic case. In the 2D case the number of parameters is equal to 4. See Figures 3.9–3.14.

$$\begin{array}{l} E_{11}(1) = 2\text{x}10^6, \ E_{33}(1) = 5\text{x}10^6, \\ E_{11}(2) = 3\text{x}10^6, \ E_{33}(2) = 9\text{x}10^6, \\ E_{11}(0) = 1\text{x}10^6, \ E_{33}(0) = 3\text{x}10^6, \\ \mu_{13}(1) = 5000, \quad \mu_{13}(2) = 5500, \\ \mu_{13}(0) = 2500. \end{array}$$

In this example the first parameter, $E_{11}(i)$ (the x_1-uniaxial modulus), is the same as E_i in the isotropic case. Here we write C_i^j in terms of E_{ii},ν_{ij} and μ_{ij} which is given implicitly by

$$C_{ij}^{-1} = \begin{pmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & -\frac{\nu_{13}}{E_{11}} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{23}}{E_{22}} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_{33}} & -\frac{\nu_{32}}{E_{33}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\mu_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\mu_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_{12}} \end{pmatrix}$$

and

$$\varepsilon_{ij} = C_{ij}^{-1}\sigma_{ij}.$$

The observation of the numerical results shows that the longitudinal displacements corresponding to the first case (when a horizontal force is applied to the upper surface) have the same order of magnitude both for isotropic and orthotropic structures.

The vertical displacement is much smaller for the orthotropic structure with the large values of E_{33}. Also all displacement components for this orthotropic structure are quite small (compared with the isotropic case) when the vertical load is applied.

3.5.1 Summary

In this chapter we have developed an efficient asymptotic algorithm for analysis of elasticity problems posed in layered structures with imperfect interfaces. The example treated in this Chapter models an adhesive joint in anisotropic stratified media. The advantage of the present asymptotic approach is that it enables one to reduce the original complicated three-dimensional problem of elasticity, posed in a region with a thin adhesive, to a set of differential equations on the limit surface. This derivation does not require any additional physical assumptions. Numerical experiments presented at the end of this chapter demonstrate the potential for using the derived equations in large scale numerical computations. The limit equations here derived are valid within the three dimensional region but near the edges they give an error that needs to be corrected. This could be an important extension of this project. We would like to mention papers relevant to this effect: Zorin and Nazarov (1989) on boundary layer for an elastic thin three-dimensional plate, Miller and Horgan (1995) on end effects for plane deformations of elastic composites and Wijeyewickrema *et al.* (1996) on end effects for plane deformations of sandwich strips.

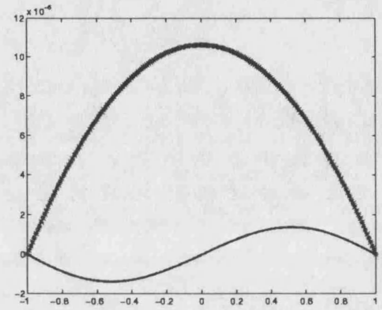

Figure 3.3: shows $v^{(1)}$ for the isotropic case. (***) Loading Case 1 (—)Loading Case 2 Amplified 100 times.

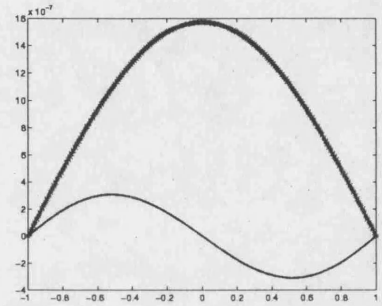

Figure 3.4: shows $v^{(2)}$ for the isotropic case. (***) Loading Case 1 (—) Loading Case 2 Amplified 5 times.

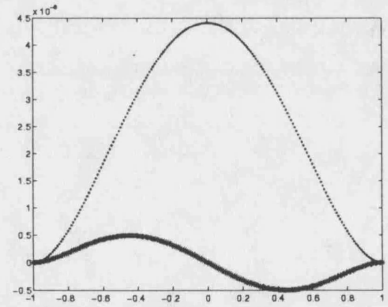

Figure 3.5: shows $u_3^{(0)}$ for the isotropic case. (***) Loading Case 1 Amplified 10 times (...) Loading Case 2.

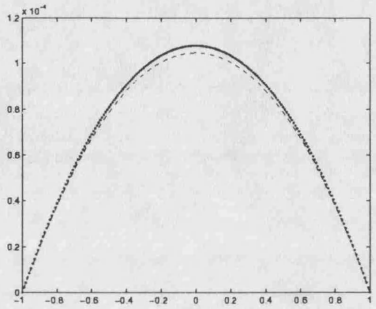

Figure 3.6: shows $v^{(1)}$ for the isotropic case . (—) Loading Case 3 (...) Loading Case 4.

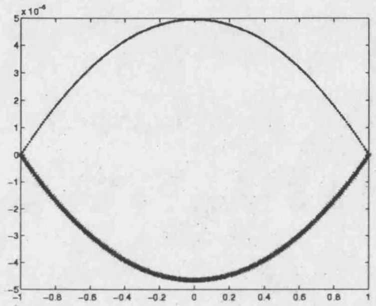

Figure 3.7: shows $v^{(2)}$ for the isotropic case . (***) Loading Case 3 (...) Loading Case 4.

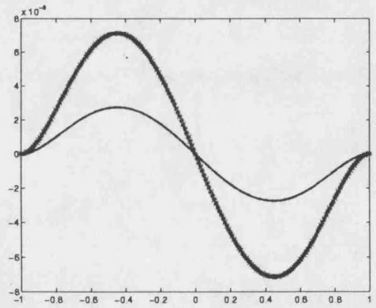

Figure 3.8: shows $u_3^{(0)}$ for the isotropic case . (***) Loading Case 3 Amplified 10 times (—) Loading Case 4.

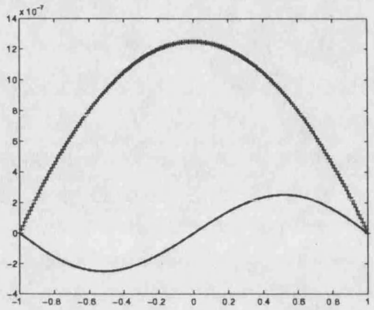

Figure 3.9: shows $v^{(1)}$ for the orthotropic case. (+++) Loading Case 1 Reduced 100 times (—) Loading Case 2 Amplified 100 times.

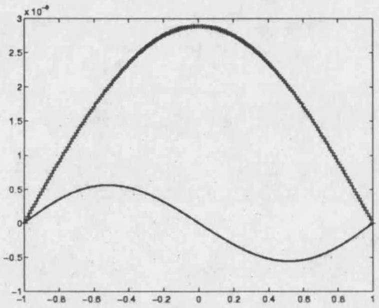

Figure 3.10: shows $v^{(2)}$ for the orthotropic case. (+++) Loading Case 1 (—) Loading Case 2 Amplified 5 times.

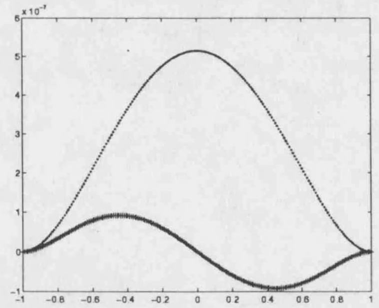

Figure 3.11: shows $u_3^{(0)}$ for the orthotropic case . (+++) Loading Case 1 Amplified 100 times (...) Loading Case 2 Reduced 10 times.

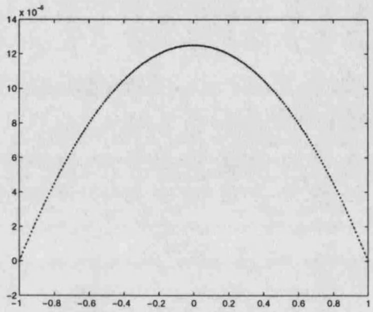

Figure 3.12: shows $v^{(1)}$ for the orthotropic case. (—) Loading Case 3 (...) Loading Case 4.

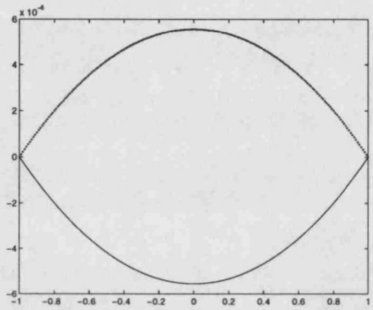

Figure 3.13: shows $v^{(2)}$ for the orthotropic case . (—) Loading Case 3 (...) Loading Case 4.

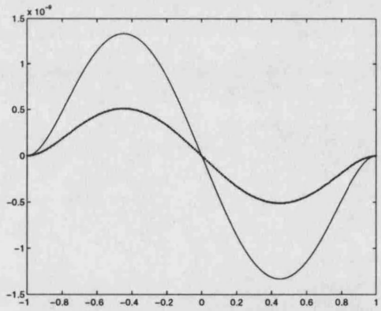

Figure 3.14: shows $u_3^{(0)}$ for the orthotropic case. (—) Loading Case 3 (...) Loading Case 4.

Chapter 4

A long-wave dynamic model for a layered structure with an adhesive joint

In this chapter a model of wave propagation in a layered structure involving a thin and soft interface is studied. The asymptotic method (see Chapters 1, 2 and 3) is applied to derive an effective contact condition. This is the condition of an imperfect interface across the thin and soft layer. The use of the asymptotic method allows one to obtain a *lower* dimensional formulation describing propagation of waves associated with the displacement jump across the interface. The vibrational response of plates in vacuo and vibrations of multilayered and sandwiched beams of similar thickness have been studied by Sorokin (1995) and Maheri and Adams (1998).

4.1 Limit equations for thin domains with dynamics

Here, the limit equations of linear isotropic elasticity in a thin region are derived using the asymptotic method. For the leading order terms of the asymptotic approximation of the displacement field, the differential equations (that correspond to a bending or a longitudinal deformation) of a thin *isotropic* beam are derived.

4.1.1 Plane strain state within a thin region

We consider a thin region Γ_ϵ of uniform thickness ϵ. The isotropic response of the material is described by the Lamé constants λ and μ. Let the displacement vector

$\boldsymbol{u}$, defined as $\boldsymbol{u} = (u_1, u_2)^T$, satisfy the Lamé system of equations of equilibrium

$$\mu\Delta \boldsymbol{u}(\epsilon, \boldsymbol{x}, \tau) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}(\epsilon, \boldsymbol{x}, \tau) = \rho\partial_\tau^2 \boldsymbol{u}(\epsilon, \boldsymbol{x}, \tau), \boldsymbol{x} \in \Omega_\epsilon, \qquad (4.1.1)$$

with free-boundary conditions

$$\sigma^{(2)}(\boldsymbol{u}; x_1, \pm\epsilon/2) = 0, |x_1| < l. \qquad (4.1.2)$$

The edges of the layered structure are assumed to be fixed:

$$\boldsymbol{u}(\pm l, x_2) = \mathbf{0}.$$

We take into account the variable t as first given in Relationship (2.1.2), with the observation that

$$t \in [-1/2, 1/2]$$

and Relationship (2.1.4).

4.1.2 The formal asymptotic expansion

We will assume that the transverse displacement component depends on the fast time variable $T = \epsilon\tau$ and the longitudinal component depends on the slow time variable τ. Let the displacement field be approximated by

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t, \tau) \sim \; & \epsilon^{-2}\sum_{q=0}^{3} \epsilon^q \boldsymbol{v}^{(q)}(x_1, t, \epsilon\tau) \\ & + \sum_{q=0}^{1} \epsilon^q \boldsymbol{V}^{(q)}(x_1, t, \tau) + \epsilon^2 \boldsymbol{W}. \end{aligned} \qquad (4.1.3)$$

The first two sums give the solvability for the problem with respect to $\boldsymbol{W} = (w_1, w_2)^T$ obtained after substitution into the equations and the zero-traction boundary conditions. These solvability conditions for w_1, w_2 constitute a well-posed system including the leading order component.

We discuss the leading ansatz of the expansion (4.1.3) and derive the differential equations for the principal part of the displacement field. These differential equations will be the equations of an elastic *isotropic* beam. Direct substitution of (4.1.3) into the system (4.1.1) and boundary conditions (4.1.2) and equating the coefficients of powers of ϵ yields the following set of boundary value problems on the cross section $[-1/2, 1/2]$.

Step 1 For $v_1^{(0)}$ the following BVP on the cross-section holds for the thin structure,

$$\begin{aligned} \partial_t^2 v_1^{(0)} &= 0, \quad \text{in } \Omega_\epsilon, \\ \partial_t v_1^{(0)} &= 0, \quad \text{on } \Gamma_+, \\ \partial_t v_1^{(0)} &= 0, \quad \text{on } \Gamma_-. \end{aligned} \tag{4.1.4}$$

Hence, the first term of the approximation (4.1.3), is a function of x_1 and T only. The solvability condition of the BVP (4.1.4) is identically satisfied. Similarly to Section 1.4.4, we assumed that

$$v_1^{(0)} \equiv 0. \tag{4.1.5}$$

For the second component $v_2^{(0)}$, the following BVP holds on the cross-section for the thin structure,

$$\begin{aligned} \partial_t^2 v_2^{(0)} &= 0, \quad \text{in } \Omega_\epsilon, \\ \partial_t v_2^{(0)} &= 0, \quad \text{on } \Gamma_+, \\ \partial_t v_2^{(0)} &= 0, \quad \text{on } \Gamma_-. \end{aligned} \tag{4.1.6}$$

Hence, the first term $v_2^{(0)}$ of the approximation (4.1.3), is a function of x_1 and T only. The solvability condition of the BVP (4.1.6) is identically satisfied.

Step 2 For the second component $v_2^{(1)}$ satisfies the following BVP on the cross-section

$$\begin{aligned} \partial_t^2 v_2^{(1)} &= 0, \quad \text{in } \Omega_\epsilon, \\ \partial_t v_2^{(1)} &= 0, \quad \text{on } \Gamma_+, \\ \partial_t v_2^{(1)} &= 0, \quad \text{on } \Gamma_-. \end{aligned} \tag{4.1.7}$$

Hence, the second term of the approximation (4.1.3) for the transverse component, is a function of x_1 and T only. Moreover, the solvability condition of the BVP (4.1.7) is identically satisfied. Similarly to Section 1.4.4 it is assumed that $v_2^{(1)}$ is zero

$$v_2^{(1)} \equiv 0.$$

For the first component $v_1^{(1)}$, the following BVP on the cross-section holds for the thin structure,

$$\begin{aligned} \partial_t^2 v_1^{(1)} &= 0, && \text{in } \Omega_\epsilon, \\ \partial_t v_1^{(1)} &= -\partial_1 v_2^{(0)}, && \text{on } \Gamma_+, \\ \partial_t v_1^{(1)} &= -\partial_1 v_2^{(0)}, && \text{on } \Gamma_-. \end{aligned} \tag{4.1.8}$$

Hence, the solvability condition for the BVP (4.1.8) is identically satisfied. Moreover, the representation for $v_1^{(1)}$ is given by

$$v_1^{(1)} = -t\partial_1 v_2^{(0)}.$$

Step 3 For $U_1^{(1)}$ we have the following BVP on the cross-section

$$\begin{aligned} \partial_t^2 U_1^{(1)} &= 0, && \text{in } \Omega_\epsilon, \\ \partial_t U_1^{(1)} &= 0, && \text{on } \Gamma_+, \\ \partial_t U_1^{(1)} &= 0, && \text{on } \Gamma_-, \end{aligned} \tag{4.1.9}$$

where $U_1^{(1)} = \mathcal{V}_1^{(0)} + v_1^{(2)}$. The solvability condition for the BVP (4.1.9) is identically satisfied. Hence, the representation for $U_1^{(1)}$ is given in terms of x_1, τ and T only. Also it can assumed that the function $v_1^{(2)}$ is zero

$$v_1^{(2)} \equiv 0.$$

Hence, the function $\mathcal{V}_1^{(0)}$ is a function of x_1 and τ only. Also for the second components of the displacement vector we have the following BVP on the cross-section

$$\begin{aligned} \partial_t^2 U_2^{(1)} &= \frac{\lambda}{2\mu+\lambda}\partial_1^2 v_2^{(0)}, && \text{in } \Omega_\epsilon, \\ \partial_t U_2^{(1)} &= \frac{\lambda}{2\mu+\lambda} t\partial_1^2 v_2^{(0)}, && \text{on } \Gamma_+, \\ \partial_t U_2^{(1)} &= \frac{\lambda}{2\mu+\lambda} t\partial_1^2 v_2^{(0)}, && \text{on } \Gamma_-, \end{aligned} \tag{4.1.10}$$

where $U_2^{(1)} = \mathcal{V}_2^{(0)} + v_2^{(2)}$. While, the solvability condition for the BVP (4.1.10) is identically satisfied, the representation for $U_2^{(1)}$ is given by

$$U_2^{(1)} = \frac{\lambda}{2\mu+\lambda} t^2 \partial_1^2 v_2^{(0)}.$$

Step 4 For $U_1^{(2)}$ we have the following BVP on the cross-section

$$
\begin{aligned}
\partial_t^2 U_1^{(2)} &= \frac{3\lambda + 4\mu}{2\mu + \lambda} t \partial_1^3 v_2^{(0)}, && \text{in } \Omega_\epsilon, \\
\partial_t U_1^{(2)} &= -\frac{\lambda}{2(2\mu + \lambda)} t^2 \partial_1^3 v_2^{(0)}, && \text{on } \Gamma_+, \\
\partial_t U_1^{(2)} &= -\frac{\lambda}{2(2\mu + \lambda)} t^2 \partial_1^3 v_2^{(0)}, && \text{on } \Gamma_-,
\end{aligned}
\tag{4.1.11}
$$

where $U_1^{(2)} = v_1^{(3)} + \mathcal{V}_1^{(1)}$. The solvability condition for the BVP (4.1.11) is identically satisfied. Moreover, one can see that

$$
U_1^{(2)} = \frac{3\lambda + 4\mu}{6(2\mu + \lambda)} t^3 \partial_1^3 v_2^{(0)} - \frac{\lambda + \mu}{2(2\mu + \lambda)} t \partial_1^3 v_2^{(0)}.
$$

Similarly we have the following BVP on the cross-section

$$
\begin{aligned}
\partial_t^2 U_2^{(2)} &= 0, && \text{in } \Omega_\epsilon, \\
\partial_t U_2^{(2)} &= -\frac{\lambda}{2\mu + \lambda} \partial_1 \mathcal{V}_1^{(0)}, && \text{on } \Gamma_+, \\
\partial_t U_2^{(2)} &= -\frac{\lambda}{2\mu + \lambda} \partial_1 \mathcal{V}_1^{(0)}, && \text{on } \Gamma_-,
\end{aligned}
\tag{4.1.12}
$$

for $U_2^{(2)} = v_2^{(3)} + \mathcal{V}_2^{(1)}$. It follows immediately that the solvability condition for the BVP (4.1.12) is identically satisfied, whereas the function $U_2^{(2)}$ is given by

$$
U_2^{(2)} = -\frac{\lambda}{2\mu + \lambda} t \partial_1 \mathcal{V}_1^{(0)}.
$$

Step 5 Here, the limit equations for $\boldsymbol{W} = (w_1, w_2)^T$ are derived. Firstly for w_1 it is seen that the following BVP on the cross-section holds,

$$
\begin{aligned}
\partial_t^2 w_1 &= -\frac{4\mu + 3\lambda}{2\mu + \lambda} \partial_1^2 \mathcal{V}_1^{(0)} + \frac{\rho}{\mu} \partial_\tau^2 \mathcal{V}_1^{(0)}, && \text{in } \Omega_\epsilon, \\
\partial_t w_1 &= \frac{\lambda}{2\mu + \lambda} t \partial_1^2 \mathcal{V}_1^{(0)}, && \text{on } \Gamma_+, \\
\partial_t w_1 &= \frac{\lambda}{2\mu + \lambda} t \partial_1^2 \mathcal{V}_1^{(0)}, && \text{on } \Gamma_-.
\end{aligned}
$$

Thus, the last BVP has the following solvability condition

$$
\boxed{\frac{4(\lambda + \mu)}{2\mu + \lambda} \partial_1^2 \mathcal{V}_1^{(0)} = \frac{\rho}{\mu} \partial_\tau^2 \mathcal{V}_1^{(0)}.}
\tag{4.1.13}
$$

This limit equation, the wave equation modelling longitudinal waves in a homogeneous beam has been derived and studied in monographs by Sorokin (1995); Achenbach (1973) and Graff (1975).

In a similar way, w_2 satisfies the following BVP on the cross-section,

$$\begin{aligned}
\partial_t^2 w_2 &= \frac{9\lambda^2 + 8\mu^2 + 16\lambda\mu}{24(2\mu+\lambda)^2}\partial_1^4 v_2^{(0)} + \frac{\rho}{2\mu+\lambda}\partial_T^2 v_2^{(0)}, && \text{in } \Omega_\epsilon,\\
\partial_t w_2 &= \frac{9\lambda^2 + 8\lambda\mu}{24(2\mu+\lambda)^2} t\partial_1^4 v_2^{(0)}, && \text{on } \Gamma_+,\\
\partial_t w_2 &= \frac{9\lambda^2 + 8\lambda\mu}{24(2\mu+\lambda)^2} t\partial_1^4 v_2^{(0)}, && \text{on } \Gamma_-.
\end{aligned}$$

The last BVP has the following solvability condition

$$\boxed{\frac{\mu(\mu+\lambda)}{3(2\mu+\lambda)}\partial_1^4 v_2^{(0)} = -\rho\partial_T^2 v_2^{(0)}.} \tag{4.1.14}$$

The Equations (4.1.13) and (4.1.14) have been derived on the basis of the asymptotic analysis of the equilibrium equations in a thin rectangle without additional physical assumptions. This last limit equation can be generalised to the one in the literature (Achenbach, 1973; Miklowitz, 1980) for the transverse motion of a thin beam of thickness h_l given by

$$D\partial_1^4 u_2 = -\rho_l h_l \partial_T^2 u_2, \tag{4.1.15}$$

where

$$D = \frac{E_l h_l^3}{12(1-\nu_l^2)}.$$

ν, ρ_l and E_l are the properties of the homogeneous beam which denote Poisson ratio, density and Young's modulus, respectively. We immediately note this is not the wave equation form. The last equation can be used to find the phase velocity of waves travelling along the beam. To do this, we seek a solution of Equation (4.1.15) in the form

$$\tilde{u}_2(x, T) = A\exp(i\{kx_1 - \omega T\}). \tag{4.1.16}$$

Hence, Equation (4.1.16) is a solution for the Equation (4.1.15) if k and ω are related to each other by the condition

$$Dk^4 - \rho_l h_l \omega^2 = 0.$$

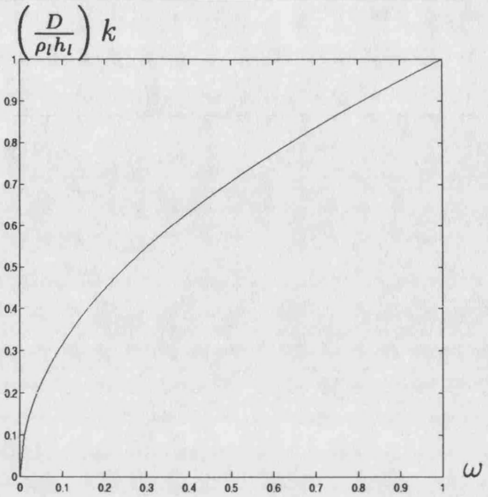

Figure 4.1: The dispersion curve.

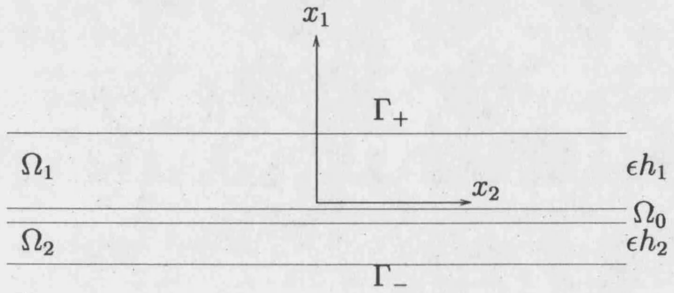

Figure 4.2: The two-dimensional thin strip Ω_ϵ.

In fact the relationship between the wave number and frequency is referred to as the dispersion relation. The last equation can be rewritten as

$$\omega = \sqrt{\frac{D}{\rho_l h_l} k^2}.$$

This relationship is illustrated by Figure 4.1.

4.2 The layered structure problem

In this section the study of propagating waves on a layered structure is studied when a signal force is applied. Again, the inner layer is made of a thin and soft material. This bonding effect has been considered in the literature (Nayfeh and

Nassar, 1978, 1982; Nayfeh, 1980b,a) without specifying explicitly this fact in the softness of the material. Consider a thin rectangular strip (see Figure 4.2), which consists of three layers:

$$\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < \infty\, , \; \epsilon(h/2 - h_1) + \epsilon^2 h_0 < x_2 < \epsilon h/2 + \epsilon^2 h_0\},$$
$$\Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < \infty\, , \; -\epsilon h/2 < x_2 < -\epsilon h/2 + \epsilon h_2\},$$
$$\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < \infty\, , \; -\epsilon(h/2 - h_2) < x_2 < -\epsilon(h/2 - h_2) + \epsilon^2 h_0\},$$

where the quantities l and h_i , $i = 0, 1, 2$, have the same order of magnitude, ϵ is a small non-dimensional positive parameter and $\boldsymbol{x} = (x_1, x_2)$ are the rectangular coordinates. We also use the notation $h = h_1 + h_2$.

The upper and lower layers have thickness ϵh_1 and ϵh_2 respectively, and the middle layer is thinner: its thickness is equal to $\epsilon^2 h_0$. The interface boundary includes two parts S_+ and S_- specified by the relationship (2.1.1). The upper and lower surfaces of the compound region are given by the relationship (2.1.2). The layers Ω_1 and Ω_2 are adhesively joined by the thin layer Ω_0, which is made of an adhesive material that is much softer than the both materials of the other two layers. Here the same non-dimensional variables given by (2.1.2) are introduced with the same observations given by relationships (2.1.3) and (2.1.4).

4.3 Anti-plane shear

In this section, a boundary value problem for the Laplacian, that corresponds to the case of an anti-plane shear of the layered structure, is studied. The displacement vector depends on x_1, x_2 and τ only, and has the form

$$\boldsymbol{u}^{(i)} = (0, 0, w^{(i)}(x_1, x_2, \tau)).$$

Considering extra-plane oscillations, we shall assume that the displacement vectors $\boldsymbol{u}^{(i)} = (0, 0, w^{(i)}(x_1, x_2, \tau))$ satisfy the equations

$$\mu_i \Delta w^{(i)}(x_1, x_2, \tau) + f_i(x_1, x_2) = \rho_i \partial_\tau^2 w^{(i)}(x_1, x_2, \tau), \quad \text{in } \Omega_i, \; i = 0, 1, 2. \tag{4.3.1}$$

We further assume that the boundary conditions on the sides $\Gamma_\pm$ are given by

$$\frac{\partial w}{\partial x_2} = p_\pm \quad \text{on } \Gamma_\pm, \tag{4.3.2}$$

where $\mu_i, i = 0, 1, 2$ are the shear moduli of the materials and $\rho_i, i = 0, 1, 2$ are the densities of the materials. To simplify our analysis it can be assumed here that $f_0 \equiv 0$. We use the superscript index notation $w^{(i)}$ to denote the displacement in the region Ω_i. The interface contact conditions on $S_\pm$ have the form

$$\mu_1 \frac{\partial w^{(1)}}{\partial x_2} = \mu_0 \frac{\partial w^{(0)}}{\partial x_2} \quad \text{on } S_+, \tag{4.3.3}$$

$$\mu_2 \frac{\partial w^{(1)}}{\partial x_2} = \mu_0 \frac{\partial w^{(0)}}{\partial x_2} \quad \text{on } S_-. \tag{4.3.4}$$

We seek the asymptotic approximation for the functions $w^{(i)}, i = 0, 1, 2$ in the form

$$\begin{aligned} w^{(i)}(x_1, x_2, \tau) \sim\ & W^{(i,0)}(x_1, \tau) + \epsilon W^{(i,1)}(x_1, t_i, \tau) \\ & + \epsilon^2 W^{(i,2)}(x_1, t_i, \tau), \qquad i = 0, 1, 2. \end{aligned} \tag{4.3.5}$$

The middle layer is normalised in such a way that

$$\mu_0 = \epsilon^3 \mu,$$

where the quantity μ has the same order of magnitude as μ_1 and μ_2. Putting the last series (4.3.5) into system (4.3.1) and equating the coefficients near like powers of the small parameter ϵ, the following recurrent system of boundary value problems on the cross-section of the compound beam can be obtained,

$$\frac{\partial^2 W^{(i,k)}}{\partial t_i^2} = -\frac{\partial^2 W^{(i,k-2)}}{\partial x_1^2} - \frac{1}{\mu_i}\left(f_i \delta_{k2} + \rho_i \partial_\tau^2 W^{(i,k-2)}\right) \quad \text{in } \Omega_i, i = 1, 2, \tag{4.3.6}$$

for the upper and lower layers and

$$\frac{\partial^2 W^{(0,k)}}{\partial t_0^2} = -\frac{\partial^2 W^{(0,k-4)}}{\partial x_1^2} + \frac{\rho_0}{\mu} \partial_\tau^2 W^{(0,k-1)} \quad \text{in } \Omega_0, \tag{4.3.7}$$

for the middle layer. Also the boundary conditions (4.3.2) become

$$\frac{\partial}{\partial t_1} W^{(1,k)} = p_+ \delta_{k1} \quad \text{on } \Gamma_+, \tag{4.3.8}$$

$$\frac{\partial}{\partial t_2} W^{(2,k)} = p_- \delta_{k1} \quad \text{on } \Gamma_-. \tag{4.3.9}$$

The interface boundaries, given by relations (4.3.3) and (4.3.4), can be written as

$$W^{(1,k)} = W^{(0,k)},\ \mu_1\partial_{t_1}W^{(1,k)} = \mu\partial_{t_0}W^{(0,k-2)} \text{ on } S_+, \tag{4.3.10}$$
$$W^{(2,k)} = W^{(0,k)},\ \mu_2\partial_{t_2}W^{(2,k)} = \mu\partial_{t_0}W^{(0,k-2)} \text{ on } S_-. \tag{4.3.11}$$

All terms with negative indices vanish. Thus, we have some recurrent sequences of the Neumann boundary value problems on the cross-section for ordinary differential equations in the upper and lower layers (the longitudinal variable x_1 is included as a parameter) and the sequence of Dirichlet boundary value problems for the middle layer. The Neumann BVPs mentioned require certain solvability conditions for the right-hand sides of the equations and boundary conditions. As a result, the limit equations for the leading terms of the approximation (4.3.5) can be found. For convenience the notation (2.2.15) shall be used throughout this chapter.

4.3.1 Analysis of the recurrent sequence for BVPs

To illustrate the technique, we consider the special case when the body force densities $f_i, i = 1, 2$ and tractions p_+ and p_- are zero.

Step 1 Substituting $k = 0$ in Relations (4.3.6) and (4.3.8)–(4.3.11) it is seen that the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial^2_{t_i}W^{(i,0)} &= 0, \text{ in } \Omega_i,\ i = 1, 2, \\ \partial_{t_i}W^{(i,0)} &= 0, \text{ on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i}W^{(i,0)} &= 0, \text{ on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{4.3.12}$$

It follows that the solvability conditions for the BVP (2.2.32) are identically satisfied. Also, with the boundary conditions, the first terms of the approximation (4.3.5), for both the upper and lower layers, are found to be functions of x_1 and τ only,

$$W^{(i,0)} \equiv W^{(i,0)}(x_1, \tau), i = 1, 2.$$

For the middle layer, Equations (4.3.7), (4.3.10) and (4.3.11) give the following

Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial^2_{t_0} W^{(0,0)} &= 0, && \text{in } \Omega_0, \\ W^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= W^{(1,0)}, && \text{on } S_+, \\ W^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= W^{(2,0)}, && \text{on } S_-. \end{aligned}$$

Therefore, the representation of $W^{(0,0)}(x_1, t_0, \tau)$ can be written in the form

$$W^{(0,0)}(x_1, t_0, \tau) = A^{(0)}(x_1, \tau) + \frac{t_0}{h_0} D^{(0)}(x_1, \tau),$$

where $A^{(0)}(x_1, \tau)$ is the average of the functions $W^{(1,0)}(x_1, \tau)$ and $W^{(2,0)}(x_1, \tau)$ evaluated at the interfaces S_+ and S_-

$$A^{(0)}(x_1, \tau) = \frac{W^{+(1,0)}(x_1, \tau) + W^{-(2,0)}(x_1, \tau)}{2}, \tag{4.3.13}$$

whereas $D^{(0)}(x_1, \tau)$ is referred to as the displacement jump evaluated at the interfaces S_+ and S_-

$$D^{(0)}(x_1, \tau) = W^{+(1,0)}(x_1, \tau) - W^{-(2,0)}(x_1, \tau). \tag{4.3.14}$$

Step 2 Making the substitution $k = 1$ in relation (4.3.6), with boundary conditions (4.3.8)–(4.3.11) it is seen that the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial^2_{t_i} W^{(i,1)} &= 0, && \text{in } \Omega_i,\ i = 1, 2, \\ \partial_{t_i} W^{(i,1)} &= 0, && \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} W^{(i,1)} &= 0, && \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{4.3.15}$$

For both BVP (4.3.15) the solvability conditions are identically satisfied as in the previous step. The representations of $W^{(i,1)}, i = 1, 2$ are given in terms of x_1 and τ only.

For the middle layer, Equations (4.3.7), (4.3.10) and (4.3.11) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial^2_{t_0} W^{(0,1)} &= 0, && \text{in } \Omega_0, \\ W^{(0,1)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= W^{(1,1)}, && \text{on } S_+, \\ W^{(0,1)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= W^{(2,1)}, && \text{on } S_-. \end{aligned}$$

Hence, the representation of $W^{(0,1)}(x_1, t_0, \tau)$ can be written in the form

$$W^{(0,1)}(x_1, t_0, \tau) = \frac{\rho_0}{\mu}\Bigg\{\partial_\tau^2 D^{(0)}(x_1,\tau)\frac{t_0^3}{6} + \partial_\tau^2 A^{(0)}(x_1,\tau)\frac{t_0^2}{2} + D^{(1)}(x_1,\tau)t_0 + A^{(1)}(x_1,\tau)\Bigg\},$$

where $A^{(1)}(x_1, \tau)$ and $D^{(1)}(x_1, \tau)$ are chosen to satisfy the following boundary conditions

$$W^{+(1,1)} = \frac{\rho_0}{\mu}\Bigg\{\partial_\tau^2 D^{(0)}(x_1,\tau)\frac{h_0^3}{48} + \partial_\tau^2 A^{(0)}(x_1,\tau)\frac{h_0^2}{8} + D^{(1)}(x_1,\tau)\frac{h_0}{2} + A^{(1)}(x_1,\tau)\Bigg\},$$

$$W^{-(2,1)} = \frac{\rho_0}{\mu}\Bigg\{-\partial_\tau^2 D^{(0)}(x_1,\tau)\frac{h_0^3}{48} + \partial_\tau^2 A^{(0)}(x_1,\tau)\frac{h_0^2}{8} - D^{(1)}(x_1,\tau)\frac{h_0}{2} + A^{(1)}(x_1,\tau)\Bigg\}.$$

Step 3 Substituting $k = 2$ in relation (4.3.6), for $i = 1$ with boundary conditions (4.3.8) and (4.3.10), it is seen that the following BVP on the cross-section holds for the upper layer,

$$\begin{aligned} &\partial_{t_1}^2 W^{(1,2)} = -\partial_1^2 W^{(1,0)} + \rho_1 \partial_\tau^2 W^{(1,0)}, && \text{in } \Omega_1,\\ &\partial_{t_1} W^{(1,2)} = 0, && \text{on } \Gamma_+,\\ &\partial_{t_1} W^{(1,2)} = \tfrac{\mu}{\mu_1}\partial_{t_0} W^{(0,0)}, && \text{on } S_+. \end{aligned} \tag{4.3.16}$$

For the BVP (4.3.16) the following solvability condition must be satisfied

$$\boxed{-\partial_1^2 W^{(1,0)}(x_1) + \frac{\rho_1}{\mu_1}\partial_\tau^2 W^{(1,0)} = -\frac{\mu}{\mu_1 h_1 h_0}\left(W^{(1,0)} - W^{(2,0)}\right).} \tag{4.3.17}$$

For $i = 2$ with $k = 2$ substituted in relation (4.3.6) with boundary conditions (4.3.9) and (4.3.11), it is seen that the following BVP on the cross-section holds for the lower layer,

$$\begin{aligned}
\partial_{t_2}^2 W^{(2,2)} &= -\partial_1^2 W^{(2,0)} + \rho_2 \partial_\tau^2 W^{(2,0)}, && \text{in } \Omega_2, \\
\partial_{t_2} W^{(2,2)} &= 0, && \text{on } \Gamma_-, \\
\partial_{t_2} W^{(2,2)} &= \frac{\mu}{\mu_2} \partial_{t_0} W^{(0,0)}, && \text{on } S_-.
\end{aligned} \tag{4.3.18}$$

For the BVP (4.3.18) the following solvability condition must be satisfied

$$\boxed{-\partial_1^2 W^{(2,0)}(x_1) + \frac{\rho_2}{\mu_2} \partial_\tau^2 W^{(2,0)} = \frac{\mu}{\mu_2 h_2 h_0} \left(W^{(1,0)} - W^{(2,0)} \right).} \tag{4.3.19}$$

4.3.2 A special dispersion relation

If one assumes that there is no displacement jump across the two layers, that is $W^{(1,0)} - W^{(2,0)} \equiv 0$, Equations (4.3.17) and (4.3.19) can be compared to the well-known model for pure shear motions governed by the following equation

$$\partial_1^2 w = \frac{\rho}{\mu} \partial_\tau^2 w,$$

where $w(x_1, \tau)$ is the anti-plane shear deformation, μ the shear modulus of the material considered and ρ its uniform density. This Equation models horizontally polarised shear motions.

Multiplying Equations (4.3.17) and (4.3.19) by $h_1 h_2$ we write these relationships in the following matrix form

$$-h_1 h_2 \partial_1^2 \boldsymbol{W}_0 + h_1 h_2 \begin{pmatrix} \frac{\rho_1}{\mu_1} & 0 \\ 0 & \frac{\rho_2}{\mu_2} \end{pmatrix} \partial_\tau^2 \boldsymbol{W}_0 + \frac{\mu}{h_0} \begin{pmatrix} \frac{h_2}{\mu_1} & -\frac{h_2}{\mu_1} \\ -\frac{h_1}{\mu_2} & \frac{h_1}{\mu_2} \end{pmatrix} \boldsymbol{W}_0 = \boldsymbol{0}, \tag{4.3.20}$$

where

$$\boldsymbol{W}_0 = \begin{pmatrix} W^{(1,0)} \\ W^{(2,0)} \end{pmatrix}.$$

In terms of the Fourier transforms

$$\bar{W}^{(i,0)} = \int_{-\infty}^{\infty} W^{(i,0)} \exp(i\omega\tau) d\tau,$$

the matrix Equation (4.3.20) can be written in the following form

$$-h_1 h_2 \partial_1^2 \bar{\boldsymbol{W}}_0(x_1) + \left\{ -\omega^2 h_1 h_2 \begin{pmatrix} \frac{\rho_1}{\mu_1} & 0 \\ 0 & \frac{\rho_2}{\mu_2} \end{pmatrix} + \frac{\mu}{h_0} \begin{pmatrix} \frac{h_2}{\mu_1} & -\frac{h_2}{\mu_1} \\ -\frac{h_1}{\mu_2} & \frac{h_1}{\mu_2} \end{pmatrix} \right\} \bar{\boldsymbol{W}}_0(x_1) = \boldsymbol{0},$$

which has the following characteristic equation

$$h_1^2 h_2^2 \xi^4 - h_1 h_2 \xi^2 \left\{ -\omega^2 h_1 h_2 \left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2} \right) + \frac{h_2 \mu}{h_0 \mu_1} + \frac{h_1 \mu}{h_0 \mu_2} \right\} - \frac{\mu^2 h_1 h_2}{\mu_1 \mu_2 h_0^2} = 0. \tag{4.3.21}$$

For convenience the following quantities are introduced

$$q = -\left\{ \frac{h_2 \mu}{h_0 \mu_1} + \frac{h_1 \mu}{h_0 \mu_2} - \omega^2 h_1 h_2 \left(\frac{1}{b_1^2} + \frac{1}{b_2^2} \right) \right\},$$
$$b_1^2 = \frac{\mu_1}{\rho_1},$$
$$b_2^2 = \frac{\mu_2}{\rho_2},$$

where b_1 and b_2 are the shear wave speeds for the upper and lower regions, respectively. Thus, the characteristic Equation (4.3.21) can be rewritten in the form

$$\xi^4 + \xi^2 \frac{q}{h_1 h_2} - \frac{\mu^2}{\mu_1 \mu_2 h_0^2 h_1 h_2} = 0. \tag{4.3.22}$$

It is shown that for any positive value of ω, the characteristic Equation (4.3.22) has pure imaginary roots corresponding to oscillation of the shear displacement in the longitudinal variable x_1.

Next, we rewrite the system (4.3.17) and (4.3.19) with respect to the average $A^{(0)}$ (see relationship (4.3.13)) and the displacement jump $D^{(0)}$ (see relationship (4.3.14)) in the following form

$$\begin{aligned} \frac{\mu}{h_0} \left(\frac{h_1}{\mu_2} - \frac{h_2}{\mu_1} \right) D^{(0)} &= h_1 h_2 \left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2} \right) \partial_\tau^2 A^{(0)} + \tfrac{h_1 h_2}{2} \left(\tfrac{\rho_1}{\mu_1} - \tfrac{\rho_2}{\mu_2} \right) \partial_\tau^2 D^{(0)} \\ &\quad -2 h_1 h_2 \partial_1^2 A^{(0)}, \\ -\frac{\mu}{h_0} \left(\frac{h_1}{\mu_2} + \frac{h_2}{\mu_1} \right) D^{(0)} &= h_1 h_2 \left(\frac{\rho_1}{\mu_1} - \frac{\rho_2}{\mu_2} \right) \partial_\tau^2 A^{(0)} + \tfrac{h_1 h_2}{2} \left(\tfrac{\rho_1}{\mu_1} + \tfrac{\rho_2}{\mu_2} \right) \partial^2 D^{(0)} \\ &\quad -h_1 h_2 \partial_1^2 D^{(0)}. \end{aligned}$$

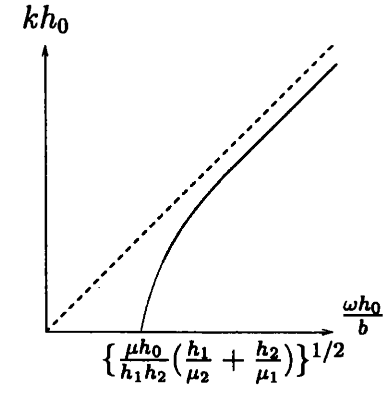

Figure 4.3: The dispersion curve.

For the special case when

$$\frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \equiv b^2,$$

the system of equations decouples and the displacement jump $D^{(0)}$ satisfies the ordinary differential equation given by

$$\partial_1^2 D^{(0)}(x_1,t) - \frac{1}{b^2}\partial_\tau^2 D^{(0)}(x_1,t) - \frac{\mu}{h_0 h_1 h_2}\left(\frac{h_1}{\mu_2}+\frac{h_2}{\mu_1}\right) D^{(0)}(x_1,t) = 0 \quad (4.3.23)$$

If we seek a solution of the Equation (4.3.23) in the form

$$D^{(0)} = A\exp(ikx_1 - i\omega\tau), \quad (4.3.24)$$

the dispersion relation is given by

$$\frac{\omega^2 h_0^2}{b^2} - k^2 h_0 = \frac{\mu h_0}{h_1 h_2}\left(\frac{h_1}{\mu_2}+\frac{h_2}{\mu_1}\right).$$

The last relationship shows the presence of the lower bound for the frequency ω (see Figure 4.3), corresponding to oscillations in the longitudinal variable x_1 for the displacement jump across the thin layer. If we compare Figures 4.3 and 4.1, we can notice that the former does not pass through the origin. In fact we see that in Figure 4.3 there is a band gap which characterises the dispersive phenomenon along the three-layered structure.

The existence of the critical frequency is shown for an anti-plane shear. If the frequency of the signal does not exceed this critical value, the oscillation of the structure is accompanied by the displacement jump which decays exponentially.

For larger values of the frequency the displacement jump waves propagate along the interface.

4.4 Plane elasticity problem

In this section, a boundary value problem for the Lamé system is studied. Here, the elastic materials of the regions $\Omega_i, i = 0, 1, 2$ are characterised by the Young's moduli E_i and by the values ν_i of the Poisson ratio, $i = 0, 1, 2$. Assume that $E_0 = \epsilon^3 E$, where E is comparable with E_1, E_2.

By λ_i, μ_i, $i = 0, 1, 2$ we denote the Lamé constants of the elastic materials first introduced in Chapter 2 (see relationship (2.4.1)). The normalised Lamé constants related to the middle layer are the same as those used in Chapter 2 (see relationship (2.4.2)).

4.4.1 Equations and Boundary Conditions

Consider extra-plane oscillations of the thin layered structure and the state of plane strain, and assume that the vectors $\boldsymbol{u}^{(i)} = (u_1, u_2)$ (the displacement field), satisfy the system

$$\mu_i \nabla^2 \boldsymbol{u}^{(i)}(\boldsymbol{x}) + (\lambda_i + \mu_i)\nabla\nabla \cdot \boldsymbol{u}^{(i)}(\boldsymbol{x}) = \rho \partial_\tau^2 \boldsymbol{u}^{(i)}, \ \boldsymbol{x} \in \Omega_i, \ i = 0, 1, 2. \quad (4.4.1)$$

For the surfaces of the compound region Ω_ϵ we prescribe free-traction conditions:

$$\begin{aligned} &\mu_1\left(\frac{\partial u_2^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_2}\right) = 0, \ (2\mu_1 + \lambda_1)\frac{\partial u_2^{(1)}}{\partial x_2} + \lambda_1 \frac{\partial u_1^{(1)}}{\partial x_1} = 0, \ \text{ on } \Gamma_+, \\ &\mu_2\left(\frac{\partial u_2^{(2)}}{\partial x_1} + \frac{\partial u_1^{(2)}}{\partial x_2}\right) = 0, \ (2\mu_2 + \lambda_2)\frac{\partial u_2^{(2)}}{\partial x_2} + \lambda_2 \frac{\partial u_1^{(2)}}{\partial x_1} = 0, \ \text{ on } \Gamma_-. \end{aligned} \quad (4.4.2)$$

The conditions of the ideal contact are specified by the Relationships (2.4.6)–(2.4.8) on the upper interface and (2.4.9)–(2.4.11) on the lower interface.

Here the same non-dimensional variables given by (2.1.2) are introduced with the same observations given by relationships (2.1.3) and (2.1.4).

4.4.2 Asymptotic approach

Here an asymptotic analysis of the dynamic elasticity problem for the thin layered strip introduced earlier is proposed. This allows one to derive some partial differential equations for the leading order components of the displacement field. We will assume that the transverse displacement component depends on the slow time variable $T = \epsilon\tau$ and the longitudinal component depends on the fast time variable τ. We assume that the displacement is approximated by

$$\boldsymbol{u}^{(i)}(\boldsymbol{x}) \sim \sum_{k=0}^{\infty} \epsilon^k \begin{pmatrix} u_1^{(i,k)}(\epsilon, x_1, t_i, \tau) \\ u_2^{(i,k)}(\epsilon, x_1, t_i, T) \end{pmatrix}, i = 0, 1, 2. \tag{4.4.3}$$

If one substitutes this series into Equations (4.4.1) and boundary conditions (4.4.2), (2.4.6)–(2.4.8) and (2.4.9)–(2.4.11) and equates the coefficients of powers of ϵ, it yields the following recurrent system of relations on the cross-section

$$\mu_i \partial_{t_i}^2 u_1^{(i,k)} + (\lambda_i + \mu_i)\partial_{t_i 1}^2 u_2^{(i,k-1)} + (\lambda_i + 2\mu_i)\partial_1^2 u_1^{(i,k-2)} = \rho_1 \partial_\tau^2 u_1^{(i,k-2)}, \tag{4.4.4}$$

$$(2\mu_i + \lambda_i)\partial_{t_i}^2 u_2^{(i,k)} + (\lambda_i + \mu_i)\partial_{t_i 1}^2 u_1^{(i,k-1)} + \mu_i \partial_1^2 u_2^{(i,k-2)} = \rho_2 \partial_T^2 u_2^{(i,k-4)}, \tag{4.4.5}$$

for $\Omega_i, i = 1, 2$ where we have used the fast time variable $T = \epsilon\tau$. For the middle layer,

$$\mu\partial_{t_0}^2 u_1^{(0,k)} + (\lambda + \mu)\partial_{t_0 1}^2 u_2^{(0,k-2)} + (\lambda + 2\mu)\partial_1^2 u_1^{(j,k-4)} = \rho_1 \partial_\tau^2 u_1^{(0,k-2)}, \tag{4.4.6}$$

$$(2\mu + \lambda)\partial_{t_0}^2 u_2^{(0,k)} + (\lambda + \mu)\partial_{t_0 1}^2 u_1^{(j,k-2)} + \mu\partial_1^2 u_2^{(0,k-4)} = \rho_2 \partial_T^2 u_2^{(0,k-4)}, \tag{4.4.7}$$

in Ω_0. As for the static case (we refer to the work of Klarbring and Movchan (1998)), we have the following interface boundaries

$$\begin{aligned} \mu_1(\partial_{t_1} u_1^{(1,k)} + \partial_1 u_2^{(1,k-1)}) &= \mu(\partial_{t_0} u_1^{(0,k-2)} + \partial_1 u_2^{(0,k-4)}), \\ (2\mu_1 + \lambda_1)\partial_{t_1} u_2^{(1,k)} + \lambda_1 \partial_1 u_1^{(1,k-1)} &= (2\mu + \lambda)\partial_{t_0} u_2^{(0,k-2)} + \lambda\partial_1 u_1^{(0,k-4)}, \\ u^{(0,k)} &= u^{(1,k)} \text{ on } S_+ \end{aligned} \tag{4.4.8}$$

and

$$\begin{aligned} \mu_2(\partial_{t_2} u_1^{(2,k)} + \partial_2 u_2^{(2,k-1)}) &= \mu(\partial_{t_0} u_1^{(0,k-2)} + \partial_1 u_2^{(0,k-4)}), \\ (2\mu_2 + \lambda_2)\partial_{t_2} u_2^{(2,k)} + \lambda_2 \partial_1 u_1^{(2,k-1)} &= (2\mu + \lambda)\partial_{t_0} u_2^{(0,k-2)} + \lambda\partial_1 u_1^{(0,k-4)}, \\ u^{(0,k)} &= u^{(2,k)} \text{ on } S_-. \end{aligned} \tag{4.4.9}$$

On the upper and lower surfaces we have

$$\begin{aligned} \mu_1(\partial_{t_1} u_1^{(1,k)} + \partial_1 u_2^{(1,k-1)}) &= 0, & (4.4.10)\\ (2\mu_1 + \lambda_1)\partial_{t_1} u_2^{(1,k)} + \lambda_1 \partial_1 u_1^{(1,k-1)} &= 0, \text{ on } \Gamma_+, \\ \mu_2(\partial_{t_2} u_1^{(2,k)} + \partial_1 u_2^{(2,k-1)}) &= 0, & (4.4.11)\\ (2\mu_2 + \lambda_2)\partial_{t_2} u_2^{(2,k)} + \lambda_2 \partial_1 u_1^{(2,k-1)} &= 0, \text{ on } \Gamma_-. \end{aligned}$$

4.4.3 Differential equations of the compound beam

Next, we analyse the solvability condition of the problems above and in that way we derive the limit equations for the leading order components of the displacement field. For further details on this technique we refer to Klarbring and Movchan (1995, 1998).

Step 1 If $k = 0$ in Equations (4.4.4) and (4.4.8)–(4.4.11), it is seen that the following BVPs on the cross-section hold for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_1^{(i,0)} &= 0, \text{ in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_1^{(i,0)} &= 0, \text{ on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_1^{(i,0)} &= 0, \text{ on } S_+(i = 1), S_-(i = 2). \end{aligned} \qquad (4.4.12)$$

It follows that the solvability conditions for the BVPs (4.4.12) are identically satisfied. Also, with the boundary conditions, the first terms of the approximation (4.4.3), for both the upper and lower layers, are found to be functions of x_1 and τ only. We will assume that the longitudinal displacement component is one order of magnitude smaller than the transverse one. Hence,

$$u_1^{(i,0)} \equiv 0, i = 1, 2. \qquad (4.4.13)$$

For the middle layer, Equations (4.4.6), (4.4.8) and (4.4.9) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,0)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_1^{(1,0)}, && \text{on } S_+, \\ u_1^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_1^{(2,0)}, && \text{on } S_-. \end{aligned}$$

Hence, using Equations (4.4.13), it follows that $u_1^{(0,0)}$ is zero,

$$u_1^{(0,0)} = 0.$$

When $k = 0$, Equations (4.4.5) and (4.4.8)–(4.4.11) give the following BVPs on the cross-section for the upper and lower parts of the layered structure,

$$\begin{aligned} \partial_{t_i}^2 u_2^{(i,0)} &= 0, \ \text{in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_2^{(i,0)} &= 0, \ \text{on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_2^{(i,0)} &= 0, \ \text{on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{4.4.14}$$

It follows immediately that the solvability conditions for the BVPs (4.4.14) are identically satisfied. With boundary conditions, the first transverse terms of the approximation (4.4.3), for both the upper and lower layers, are found to be functions of x_1 and T only,

$$u_2^{(i,0)} = u_2^{(i,0)}(x_1, T).$$

For the middle layer, Equations (4.4.7)–(4.4.9) give the following Dirichlet BVP on the cross-section

$$\begin{aligned} \partial_{t_0}^2 u_2^{(0,0)} &= 0, & &\text{in } \Omega_0, \\ u_2^{(0,0)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_2^{(1,0)}, & &\text{on } S_+, \\ u_2^{(0,0)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_2^{(2,0)}, & &\text{on } S_-. \end{aligned}$$

Hence, the representation of $u_2^{(0,0)}(x_1, t_0, T)$ can be written in the form

$$u_2^{(0,0)}(x_1, t_0, T) = A_v^{(0)}(x_1, T) + \frac{t_0}{h_0} D_v^{(0)}(x_1, T),$$

where $A_v^{(0)}(x_1, T)$ represents the average of the functions $u_2^{(1,0)}(x_1, T)$ and $u_2^{(2,0)}(x_1, T)$ evaluated at the interfaces S_+ and S_-

$$A_v^{(0)}(x_1, T) = \frac{u_2^{+(1,0)}(x_1, T) + u_2^{-(2,0)}(x_1, T)}{2},$$

whereas $D_v^{(0)}(x_1, T)$ is referred to as the displacement jump evaluated at the interfaces S_+ and S_-

$$D_v^{(0)}(x_1, T) = u_2^{+(1,0)}(x_1, T) - u_2^{-(2,0)}(x_1, T).$$

Here the subindex v denotes that the functions are related to the *transverse component.*

Step 2 Proceeding in the same way, for the function $u_1^{(1,1)}$ it can be verified that Equations (4.4.4), (4.4.10) and (4.4.8) give the following Neumann BVP on the cross-section for the upper layer,

$$\begin{aligned} \partial_{t_1}^2 u_1^{(1,1)} &= 0, && \text{in } \Omega_1, \\ \partial_{t_1} u_1^{(1,1)} &= -\partial_1 u_2^{(1,0)}, && \text{on } \Gamma_+, \\ \partial_{t_1} u_1^{(1,1)} &= -\partial_1 u_2^{(1,0)}, && \text{on } S_+. \end{aligned} \tag{4.4.15}$$

The solvability condition for the BVP (4.4.15) is identically satisfied.

Proceeding in the same way with $u_1^{(2,1)}$ it can be seen that the following Neumann BVP on the cross-section holds for the lower layer,

$$\begin{aligned} \partial_{t_2}^2 u_1^{(2,1)} &= 0, && \text{in } \Omega_2, \\ \partial_{t_2} u_1^{(2,1)} &= -\partial_1 u_2^{(2,0)}, && \text{on } \Gamma_-, \\ \partial_{t_2} u_1^{(2,1)} &= -\partial_1 u_2^{(2,0)}, && \text{on } S_-. \end{aligned} \tag{4.4.16}$$

The solvability condition for the BVP (4.4.16) is also identically satisfied. Thus, the representation for the leading order term of the longitudinal displacements can be represented as

$$u_1^{(i,1)} = -t_i \partial_1 u_2^{(i,0)} + v^{(i)}(x_1, \tau) \; i = 1, 2$$

and for $u_1^{(0,1)}$ the following Dirichlet BVP on the cross-section holds

$$\begin{aligned} \partial_{t_0}^2 u_1^{(0,1)} &= 0, && \text{in } \Omega_0, \\ u_1^{(0,1)} &= u_1^{(1,1)}, && \text{on } S_+, \\ u_1^{(0,1)} &= u_1^{(2,1)}, && \text{on } S_-. \end{aligned}$$

Therefore one finds that

$$u_1^{(0,1)}(x_1, t_0, \tau) = A_I^{(1)}(x_1, \tau) + \frac{t_0}{h_0} D_I^{(1)}(x_1, \tau),$$

where $A_I^{(1)}(x_1, \tau)$ represents the average of the functions $u_1^{(1,1)}(x_1, \tau)$ and $u_1^{(2,1)}(x_1, \tau)$ evaluated at the interfaces S_+ and S_-

$$A_I^{(1)}(x_1, \tau) = \frac{u_1^{+(1,1)}(x_1, \tau) + u_1^{-(2,1)}(x_1, \tau)}{2},$$

whereas $D_I^{(1)}(x_1, \tau)$ is the longitudinal displacement jump evaluated at the interfaces S_+ and S_-

$$D_I^{(1)}(x_1, \tau) = u_1^{+(1,1)}(x_1, \tau) - u_1^{-(2,1)}(x_1, \tau).$$

Here the subindex I denotes that the functions are related to the *longitudinal component.*

For k=1 Equations (4.4.5) and (4.4.8)–(4.4.11) satisfy the following Neumann BVPs on the cross-section,

$$\begin{aligned} \partial^2_{t_i} u_2^{(i,1)} &= 0, \text{ in } \Omega_i, i = 1, 2, \\ \partial_{t_i} u_2^{(i,1)} &= 0, \text{ on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\ \partial_{t_i} u_2^{(i,1)} &= 0, \text{ on } S_+(i = 1), S_-(i = 2). \end{aligned} \tag{4.4.17}$$

The solvability condition for the Neumann BVP (4.4.17) is identically satisfied and the functions $u_2^{(i,1)}, i = 1, 2$ are given in terms of x_1 and T only.

$$u_2^{(i,1)} = u_2^{(i,1)}(x_1, T).$$

For the middle layer, Equations (4.4.7)–(4.4.9) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned} \partial^2_{t_0} u_2^{(0,1)} &= 0, && \text{in } \Omega_0, \\ u_2^{(0,1)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_2^{(1,1)}, && \text{on } S_+, \\ u_2^{(0,1)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_2^{(2,1)}, && \text{on } S_-. \end{aligned}$$

Therefore, it follows that $u_2^{(0,1)}(x_1, t_0, T)$ is given by

$$u_2^{(0,1)}(x_1, t_0, T) = A_v^{(1)}(x_1, T) + \frac{t_0}{h_0} D_v^{(1)}(x_1, T).$$

where $A_v^{(1)}(x_1, T)$ is the average of the functions $u_2^{(1,1)}$ and $u_2^{(2,1)}$ evaluated at the interfaces S_+ and S_-

$$A_v^{(1)}(x_1) = \frac{u_2^{+(1,1)}(x_1, T) + u_2^{-(2,1)}(x_1, T)}{2},$$

whereas $D_v^{(1)}(x_1, T)$ is the displacement jump evaluated at the interfaces S_+ and

S_-

$$D_v^{(1)}(x_1) = u_2^{+(1,1)}(x_1, T) - u_2^{-(2,1)}(x_1, T). \tag{4.4.18}$$

Here the subindex T denotes that the functions are related to the *transverse component.*

Step 3 Writing (4.4.4) for $k = 2$, and with boundary conditions (4.4.8)–(4.4.11) it is seen that the following Neumann BVPs on the cross-section can be established,

$$\begin{aligned}
\partial_{t_i}^2 u_1^{(i,2)} &= 0, \text{ in } \Omega_i, i = 1, 2, \\
\partial_{t_i} u_1^{(i,2)} &= -\partial_1 u_2^{(i,1)}, \text{ on } \Gamma_+(i = 1), \Gamma_-(i = 2), \\
\partial_{t_i} u_1^{(i,2)} &= -\partial_1 u_2^{(i,1)}, \text{ on } S_+(i = 1), S_-(i = 2).
\end{aligned} \tag{4.4.19}$$

The solvability conditions for the Neumann BVPs (4.4.19) are identically satisfied. The solution for $u_1^{(i,2)}, i = 1, 2$ gives

$$u_1^{(i,2)} = u_1^{(i,2)}(x_1, \tau) = -t_i u_2^{(i,1)}.$$

For the middle layer, Equations (4.4.6), (4.4.8) and (4.4.9) give the following Dirichlet BVP on the cross-section,

$$\begin{aligned}
\partial_{t_0}^2 u_1^{(0,2)} &= 0, && \text{in } \Omega_0, \\
u_1^{(0,2)}\left(x_1, t_0 = \tfrac{h_0}{2}\right) &= u_1^{(1,2)}, && \text{on } S_+, \\
u_1^{(0,2)}\left(x_1, t_0 = -\tfrac{h_0}{2}\right) &= u_1^{(2,2)}, && \text{on } S_-.
\end{aligned}$$

Therefore, it follows that $u_1^{(0,2)}(x_1, t_0, \tau)$ is given by

$$u_1^{(0,2)}(x_1, t_0, \tau) = A_I^{(2)}(x_1, \tau) + \frac{t_0}{h_0} D_I^{(2)}(x_1, \tau), \tag{4.4.20}$$

where $A_I^{(2)}(x_1, \tau)$ is the average of the functions $u_1^{(1,2)}(x_1, \tau)$ and $u_1^{(2,2)}(x_1, \tau)$ evaluated at the interfaces S_+ and S_-

$$A_I^{(2)}(x_1, \tau) = \frac{u_1^{+(1,2)}(x_1, \tau) + u_1^{-(2,2)}(x_1, \tau)}{2},$$

whereas $D_I^{(2)}(x_1, \tau)$ is the longitudinal displacement jump evaluated at the interfaces S_+ and S_- and is given by

$$D_I^{(2)}(x_1,\tau) = u_1^{+(1,2)}(x_1,\tau) - u_1^{-(2,2)}(x_1,\tau).$$

Proceeding with the transverse components, from steps 1 and 2, we obtain that the following Neumann BVPs on the cross-section hold for $u_2^{(i,2)}, i = 1,2,$

$$\begin{aligned}
\partial_{t_i}^2 u_2^{(i,2)} &= \frac{\lambda_i}{2\mu_i + \lambda_i}\partial_1^2 u_2^{(i,0)}, && \text{in } \Omega_i,\ i = 1,2,\\
\partial_{t_i} u_2^{(i,2)} &= -\frac{\lambda_i}{2\mu_i + \lambda_i}\left\{ -t_i\partial_1^2 u_2^{(i,0)} + \partial_1 v^{(i)} \right\}, && \text{on } \Gamma_+, (i=1), \Gamma_-, (i=2),\\
\partial_{t_i} u_2^{(i,2)} &= -\frac{\lambda_i}{2\mu_i + \lambda_i}\left\{ -t_i\partial_1^2 u_2^{(i,0)} + \partial_1 v^{(i)} \right\} \\
&\quad + \frac{2\mu + \lambda}{2\mu_i + \lambda_i}\partial_{t_0} u_2^{(0,0)}, && \text{on } S_+, (i=1), S_-, (i=2).
\end{aligned}$$

The solvability conditions for these problems are given by

$$\partial_{t_0} u_2^{(0,0)} = \frac{D_v^{(0)}}{h_0} = 0 \tag{4.4.21}$$

and so, using the Equation (4.4.15) the following notation is introduced

$$u_2^{(1,0)} = u_2^{(2,0)} = u_2^{(0,0)} \equiv u_2^{(0)}.$$

Taking into account the solvability condition (4.4.21), the solution for $u_2^{(i,2)}$ is,

$$u_2^{(i,2)} = \frac{\lambda_i}{2\mu_i + \lambda_i}\left[\frac{t_i^2}{2}\partial_1^2 u_2^{(0)} - t_i\partial_1 v^{(i)}\right], \ i = 1,2.$$

For the middle layer, Equations (4.4.7)–(4.4.9) give the following Dirichlet BVP on the cross-section

$$\begin{aligned}
\partial_{t_0}^2 u_2^{(0,2)} &= 0, && \text{in } \Omega_0,\\
u_2^{(0,2)} &= u_2^{(1,2)}, && \text{on } S_+,\\
u_2^{(0,2)} &= u_2^{(2,2)}, && \text{on } S_-.
\end{aligned}$$

Hence, one finds that the representation of $u_2^{(0,2)}(x_1, t_0, T)$ can be written in the following form

$$u_2^{(0,2)}(x_1,t_0,T) = A_v^{(2)}(x_1,T) + \frac{t_0}{h_0} D_v^{(2)}(x_1,T),$$

where $A_v^{(2)}(x_1,T)$ represents the average of the functions $u_2^{(1,2)}(x_1,T)$ and $u_2^{(2,2)}(x_1,T)$ evaluated at the interfaces S_+ and S_-

$$A_v^{(2)}(x_1,T) = \frac{u_2^{+(1,2)}(x_1,T) + u_2^{-(2,2)}(x_1,T)}{2},$$

whereas $D_v^{(2)}(x_1,T)$ is referred to as the longitudinal displacement jump evaluated at the interfaces S_+ and S_-

$$D_v^{(2)}(x_1,T) = u_2^{+(1,2)}(x_1,T) - u_2^{-(2,2)}(x_1,T).$$

Step 4 Here we shall derive second order partial differential equations for the functions $v^{(j)}$. When k=3 one can show, similarly to the previous steps, that $u_2^{(j,3)} = 0,\ j = 1,2$ and that the functions $u_1^{(j,3)}$ satisfy the following Neumann BVP on the cross-section

$$\begin{aligned}
\partial_{t_1}^2 u_1^{(1,3)} &= \frac{3\lambda_1 + 4\mu_1}{2\mu_1 + \lambda_1}\left\{ t_1 \partial_1^3 u_2^{(0)} - \partial_1^2 v^{(1)} \right\} + \frac{\rho_1}{\mu_1}\partial_\tau^2 v^{(1)}, \quad \text{in } \Omega_1, \\
\partial_{t_1} u_1^{(1,3)} &= \frac{\mu}{\mu_1 h_0}\left\{ \frac{h_1 + h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)} \right\} \\
&\quad - \frac{\lambda_1}{2\mu_1 + \lambda_1}\left\{ \frac{h_1^2}{8}\partial_1^3 u_2^{(0)} + \frac{h_1}{2}\partial_1^2 v^{(1)} \right\}, \quad \text{on } S_+, \\
\partial_{t_1} u_1^{(1,3)} &= -\frac{\lambda_1}{2\mu_1 + \lambda_1}\left\{ \frac{h_1^2}{8}\partial_1^3 u_2^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)} \right\}, \quad \text{on } \Gamma_+.
\end{aligned} \tag{4.4.22}$$

The solvability condition for the problem (4.4.22) has the form

$$\boxed{\begin{aligned}
\frac{4(\lambda_1 + \mu_1)}{2\mu_1 + \lambda_1} h_1 \partial_1^2 v^{(1)} &= \frac{\mu}{\mu_1 h_0}\left\{ \frac{h_1 + h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)} \right\} \\
&\quad + \frac{\rho_1 h_1}{\mu_1}\partial_\tau^2 v^{(1)}.
\end{aligned}} \tag{4.4.23}$$

The function $u_1^{(1,3)}$ is given by

$$
\begin{aligned}
u_1^{(1,3)} = {} & \frac{3\lambda_1 + 4\mu_1}{2\mu_1 + \lambda_1}\left(\frac{t_1^3}{6}\partial_1^3 u_2^{(0)} - \frac{t_1^2}{2}\partial_1^2 v^{(1)}\right) + \frac{\rho_1}{\mu_1}\frac{t_1^2}{2}\partial_\tau^2 v^{(1)} \\
& -\frac{\lambda_1 + \mu_1}{2\mu_1 + \lambda_1} t_1 \left(\frac{h_1^2}{2}\partial_1^3 u_2^{(0)} - \frac{h_1}{2}\partial_1^2 v^{(1)}\right) \\
& -t_1 \frac{\rho_1}{\mu_1}\frac{h_1}{2}\partial_\tau^2 v^{(1)}.
\end{aligned}
$$

We apply the same procedure for $i = 2$ and derive

$$
\boxed{
\begin{aligned}
\frac{4(\lambda_2 + \mu_2)}{2\mu_2 + \lambda_2} h_2 \partial_1^2 v^{(2)} = {} & -\frac{\mu}{\mu_2 h_0}\left\{\frac{h_1 + h_2}{2}\partial_1 u_2^{(0)} + v^{(1)} - v^{(2)}\right\} \\
& + \frac{\rho_2 h_2}{\mu_2}\partial_\tau^2 v^{(2)}.
\end{aligned}
}
\tag{4.4.24}
$$

The function $u_1^{(2,3)}$ is given by

$$
\begin{aligned}
u_1^{(2,3)} = {} & \frac{3\lambda_2 + 4\mu_2}{2\mu_2 + \lambda_2}\left(\frac{t_2^3}{6}\partial_1^3 u_2^{(0)} - \frac{t_2^2}{2}\partial_1^2 v^{(2)}\right) + \frac{\rho_2}{\mu_2}\frac{t_2^2}{2}\partial_\tau^2 v^{(2)} \\
& -\frac{\lambda_2 + \mu_2}{2\mu_2 + \lambda_2} t_2 \left(\frac{h_2^2}{2}\partial_1^3 u_2^{(0)} + \frac{h_2}{2}\partial_1^2 v^{(2)}\right) \\
& +t_2 \frac{\rho_2}{\mu_2}\frac{h_2}{2}\partial_\tau^2 v^{(2)}.
\end{aligned}
$$

If one assumes that the upper and lower layers are symmetric, that is, if they are made of the same material and have the same thickness, one could compare Equations (4.4.23) and (4.4.24) with the single limit equation for the longitudinal displacement after adding these relationships and cancelling the displacement jump and the transverse component. The resulting equation is similar to the well-known wave equation modelling longitudinal waves in a homogeneous beam of thickness h_l given by

$$
E_l \partial_1^2 u_1 = \rho_l \partial_\tau^2 u_1, \tag{4.4.25}
$$

where u_1 denotes the longitudinal displacement component, E_l the Young's modulus of the beam and ρ_l its density.

Writing (4.4.5) for $k = 3$ with boundary conditions (4.4.8) and (4.4.10) it is seen that the following Neumann BVP on the cross-section can be established,

$$
\begin{aligned}
\partial^2_{t_1} u_2^{(1,3)} &= -\frac{\mu_1}{\lambda_1 + 2\mu_1}\partial_1^2 u_2^{(1,1)}, \text{ in } \Omega_1,\\
\partial_{t_1} u_2^{(1,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(1,2)}, \text{ on } \Gamma_+,\\
\partial_{t_1} u_2^{(1,3)} &= -\frac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(1,2)} + \frac{\lambda + 2\mu}{\lambda_1 + 2\mu_1}\partial_{t_0} u_2^{(0,1)}, \text{ on } S_+.
\end{aligned} \tag{4.4.26}
$$

The solvability condition of the Neumann BVP (4.4.26) is given by

$$
(\mu_1 + \lambda_1) h_0 h_1 \partial_1^2 u_2^{(1,1)}(x_1) = (\lambda + 2\mu) D_T^{(1)}(x_1). \tag{4.4.27}
$$

Proceeding in similar way with the lower layer, we have that the following Neumann BVP on the cross-section is satisfied

$$
\begin{aligned}
\partial^2_{t_2} u_2^{(2,3)} &= -\frac{\mu_2}{\lambda_2 + 2\mu_2}\partial_1^2 u_2^{(2,1)}, \text{ in } \Omega_2,\\
\partial_{t_2} u_2^{(2,3)} &= -\frac{\lambda_2}{\lambda_2 + 2\mu_2}\partial_1 u_1^{(2,2)}, \text{ on } \Gamma_-,\\
\partial_{t_2} u_2^{(2,3)} &= -\tfrac{\lambda_1}{\lambda_1 + 2\mu_1}\partial_1 u_1^{(1,2)} + \tfrac{\lambda + 2\mu}{\lambda_1 + 2\mu_1}\partial_{t_0} u_2^{(0,1)}, \text{ on } S_-.
\end{aligned} \tag{4.4.28}
$$

The solvability condition of the last Neumann BVP (4.4.28) is given by

$$
(\mu_2 + \lambda_2) h_0 h_2 \partial_1^2 u_2^{(2,1)}(x_1) = -(\lambda + 2\mu) D_T^{(1)}(x_1). \tag{4.4.29}
$$

Step 5 Writing (4.4.5) for $k = 4$ and with boundary conditions (4.4.8) and (4.4.10) it can be seen that the following Neumann BVP on the cross-section can be established,

$$
\begin{aligned}
\partial^2_{t_1} u_2^{(1,4)} &= \frac{\rho_1}{2\mu_1+\lambda_1}\partial_T^2 u_2^{(0)} + \frac{2\mu_1+3\lambda_1}{2\mu_1+\lambda_1}\left[t_1\partial_1^3 v^{(1)} - \frac{t_1^2}{2}\partial_1^4 u_2^{(0)}\right] \\
&\quad + \frac{4(\lambda_1+\mu_1)^2}{(2\mu_1+\lambda_1)^2}\left[\frac{h_1^2}{8}\partial_1^4 u_2^{(0)} - \frac{h_1}{2}\partial_1^3 v^{(1)}\right] \\
&\quad \frac{(\lambda_1+\mu_1)\rho_1}{\mu_1(2\mu_1+\lambda_1)}\left(\frac{h_1}{2} - t_1\right)\partial^3_{1\tau\tau} v^{(1)}, && \text{in } \Omega_1, \\
\partial_{t_1} u_2^{(1,4)} &= \frac{9\lambda_1^2+8\mu_1\lambda_1}{48(2\mu_1^2+\lambda_1)^2} h_1^3\partial_1^4 u_2^{(0)} - \frac{\lambda_1(\lambda_1+\mu_1)}{(2\mu_1+\lambda_1)^2} h_1^2\partial_1^3 v^{(1)} \\
&\quad + \frac{\lambda_1\rho_1}{4\mu_1(2\mu_1+\lambda_1)} h_1^2\partial^3_{1\tau\tau} v^{(1)}, && \text{on } \Gamma_+, \\
\partial_{t_1} u_2^{(1,4)} &= -\frac{9\lambda_1^2+8\mu_1\lambda_1}{48(2\mu_1^2+\lambda_1)^2} h_1^3\partial_1^4 u_2^{(0)} + \frac{\lambda_1(\lambda_1+\mu_1)}{(2\mu_1+\lambda_1)^2} h_1^2\partial_1^3 v^{(1)} \\
&\quad - \frac{\lambda_1\rho_1}{4\mu_1(2\mu_1+\lambda_1)} h_1^2\partial_{1\tau\tau} v^{(1)} + \frac{2\mu+\lambda}{2\mu_1+\lambda_1}\partial_{t_0} u_2^{(0,2)}, && \text{on } S_+.
\end{aligned}
\tag{4.4.30}
$$

The BVP (4.4.30) is solvable if and only if:

$$
\begin{aligned}
-\frac{1}{3}\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1} h_1^3\partial_1^4 u_2^{(0)} + 2\mu_1\frac{\lambda_1+\mu_1}{2\mu_1+\lambda_1} h_1^2\partial_1^3 v^{(1)} &= (2\mu+\lambda)\partial_{t_0} u_2^{(0,2)}(S_+) \\
&\quad + \frac{h_1^2}{2}\partial^4_{11\tau\tau} v^{(1)} + \rho_1 h_1 \partial_T^2 u_2^{(0)}.
\end{aligned}
\tag{4.4.31}
$$

Also for the lower layer it can be verified that the following BVP on the cross-section holds,

$$
\begin{aligned}
\partial_{t_2}^2 u_2^{(2,4)} &= \frac{\rho_2}{2\mu_2+\lambda_2}\partial_T^2 u_2^{(0)} + \frac{2\mu_2+3\lambda_2}{2\mu_2+\lambda_2}\left[t_2\partial_1^3 v^{(2)} - \frac{t_2^2}{2}\partial_1^4 u_2^{(0)}\right] \\
&+\frac{4(\lambda_2+\mu_2)^2}{(2\mu_2+\lambda_2)^2}\left[\frac{h_2^2}{8}\partial_1^4 u_2^{(0)} + \frac{h_2}{2}\partial_1^3 v^{(2)}\right] \\
&\frac{(\lambda_2+\mu_2)\rho_2}{\mu_2(2\mu_2+\lambda_2)}\left(-\frac{h_2}{2}-t_2\right)\partial_{1\tau\tau}^3 v^{(2)}, && \text{in } \Omega_2, \\
\partial_{t_2} u_2^{(2,4)} &= -\frac{9\lambda_2^2+8\mu_2\lambda_2}{48(2\mu_2^2+\lambda_2)^2}h_2^3\partial_1^4 u_2^{(0)} - \frac{\lambda_2(\lambda_2+\mu_2)}{(2\mu_2+\lambda_2)^2}h_2^2\partial_1^3 v^{(2)} && (4.4.32) \\
&+\frac{\lambda_2\rho_2}{4\mu_2(2\mu_2+\lambda_2)}h_2^2\partial_{1\tau\tau}^3 v^{(2)}, && \text{on } \Gamma_-, \\
\partial_{t_2} u_2^{(2,4)} &= \frac{9\lambda_2^2+8\mu_2\lambda_2}{48(2\mu_2+\lambda_2)^2}h_2^3\partial_1^4 u_2^{(0)} + \frac{\lambda_2(\lambda_2+\mu_2)}{(2\mu_2+\lambda_2)^2}h_2^2\partial_1^3 v^{(2)} \\
&-\frac{\lambda_2\rho_2}{4\mu_2(2\mu_2+\lambda_2)}h_2^2\partial_{1\tau\tau} v^{(2)} + \frac{2\mu+\lambda}{2\mu_2+\lambda_2}\partial_{t_0} u_2^{(0,2)}, && \text{on } S_-.
\end{aligned}
$$

The BVP (4.4.32) is solvable if and only if:

$$
\begin{aligned}
\frac{1}{3}\frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\partial_1^4 u_2^{(0)} + 2\mu_2\frac{\lambda_2+\mu_2}{2\mu_2+\lambda_2}h_2^2\partial_1^3 v^{(2)} &= (2\mu+\lambda)\partial_{t_0} u_2^{(0,2)}(S_+) \\
&-\tfrac{h_2^2}{2}\partial_{11\tau\tau}^4 v^{(2)} - \rho_2 h_2 \partial_T^2 u_2^{(0)}.
\end{aligned}
\tag{4.4.33}
$$

We note that

$$\partial_T = \epsilon\partial_\tau.$$

Neglecting the derivatives with respect to T and combining Equations (4.4.31) and (4.4.33) we derive the following leading order lower-dimensional model

$$
\boxed{
\begin{aligned}
&\frac{1}{3}\left\{\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^3 + \frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\right\}\partial_1^4 u_2^{(0)} \\
&+2\left\{\frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^2\partial_1^3 v^{(2)} - \frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^2\partial_1^3 v^{(1)}\right\} \\
&\quad = -\frac{1}{2}\left\{\rho_1 h_1^2\partial_{1\tau\tau}^3 v^{(1)} - \rho_2 h_2^2 \partial_{1\tau\tau}^3 v^{(2)}\right\}.
\end{aligned}
}
\tag{4.4.34}
$$

4.4.4 Special solution

We assume that $u_2^{(0)}$ is a time independent function, that $v^{(j)}, j = 1, 2$ are written in the form

$$v^{(j)}(x_1, \tau) = u^{(j)}(x_1) + V^j(\tau) \tag{4.4.35}$$

where

$$V^j(\tau) = A_j e^{-i\omega\tau} \tag{4.4.36}$$

and A_j are real constants. Hence, the system becomes

$$\frac{4(\lambda_1 + \mu_1)}{2\mu_1 + \lambda_1} h_1 \partial_1^2 u^{(1)}(x_1) = \frac{\mu}{\mu_1 h_0} \{\frac{1}{2}(h_1 + h_2)\partial_1 u_2^{(0)}(x_1) + u^{(1)}(x_1) - u^{(2)}(x_1)\}$$
$$+ \frac{\mu}{\mu_1 h_0}(A_1 - A_2)e^{-i\omega\tau} - \frac{\rho_1 h_1}{\mu_1} A_1 \omega^2 e^{-i\omega\tau}, \tag{4.4.37}$$
$$\frac{4(\lambda_2 + \mu_2)}{2\mu_2 + \lambda_2} h_2 \partial_1^2 u^{(2)}(x_1) = -\frac{\mu}{\mu_2 h_0} \{\frac{1}{2}(h_1 + h_2)\partial_1 u_2^{(0)}(x_1) + u^{(1)}(x_1) - u^{(2)}(x_1)\}$$
$$- \frac{\mu}{\mu_2 h_0}(A_1 - A_2)e^{i\omega\tau} - \frac{\rho_2 h_2}{\mu_2} A_2 \omega^2 e^{-i\omega\tau}, \tag{4.4.38}$$
$$\frac{1}{3}\left\{\frac{\mu_1(\mu_1 + \lambda_1)}{2\mu_1 + \lambda_1} h_1^3 + \frac{\mu_2(\mu_2 + \lambda_2)}{2\mu_2 + \lambda_2} h_2^3\right\} \partial_1^4 u_2^{(0)}(x_1)$$
$$+2\left\{\frac{\mu_2(\mu_2 + \lambda_2)}{2\mu_2 + \lambda_2} h_2^2 \partial_1^3 u^{(2)}(x_1) - \frac{\mu_1(\mu_1 + \lambda_1)}{2\mu_1 + \lambda_1} h_1^2 \partial_1^3 u^{(1)}(x_1)\right\} = 0. \tag{4.4.39}$$

From Equations (4.4.37) and (4.4.38) one can easily see that the two last terms on the right hand sides do not depend on x_1 so they must vanish,

$$\boxed{\frac{\mu}{\mu_1 h_0}(A_1 - A_2)e^{-i\omega\tau} - \frac{\rho_1 h_1}{\mu_1} A_1 \omega^2 e^{-i\omega\tau} = 0,}$$

$$\boxed{-\frac{\mu}{\mu_2 h_0}(A_1 - A_2)e^{-i\omega\tau} - \frac{\rho_2 h_2}{\mu_2} A_2 \omega^2 e^{-i\omega\tau} = 0.}$$

We then look for a non-trivial solution A_1, A_2 and we require that

$$\mathcal{H} = \begin{pmatrix} \frac{\omega^2 h1}{b_1^2} - \frac{\mu}{\mu_1 h_0} & \frac{\mu}{\mu_1 h_0} \\ \frac{\mu}{\mu_2 h_0} & \frac{\omega^2 h2}{b_2^2} - \frac{\mu}{\mu_2 h_0} \end{pmatrix}$$

has zero determinant. Here $b_j^2 = \mu_j/\rho_j, j = 1,2$ denote the shear wave speeds. Direct calculation shows that ω satisfies

$$\omega^2\left(\frac{\omega^2 h_1 h_2}{b_1^2 b_2^2} - \left\{\frac{h_1\mu}{b_1^2\mu_2 h_0} + \frac{h_2\mu}{b_2^2\mu_1 h_0}\right\}\right) = 0,$$

and for the special case when

$$b_1^2 = b_2^2 = b^2,$$

the solution for ω gives

$$\omega = \frac{b}{h_0}\sqrt{\frac{\mu h_0}{h_1 h_2}\left(\frac{h_1}{\mu_2} + \frac{h_2}{\mu_1}\right)}. \tag{4.4.40}$$

We have taken only those solutions for ω which are non-trivial, otherwise we get limit equations for the static problem. If we constrain the vector $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ to be unit then we obtain

$$\hat{A} = \begin{pmatrix} -h_2\mu_2 \\ h_1\mu_1 \end{pmatrix} \frac{1}{\sqrt{h_1^2\mu_1^2 + h_2^2\mu_2^2}}. \tag{4.4.41}$$

4.4.5 The symmetric case

Here we assume that the upper and lower layers are symmetric and made of the same material. Also we will look for the special solution when the sum of the longitudinal displacement components $v^{(1)}(x_1,\tau) + v^{(2)}(x_1,\tau)$ vanishes. If we subtract Equation (4.4.38) from Equation (4.4.37), we get the following system of partial differential equations

$$\begin{aligned}
&\frac{4(\mu_1+\lambda_1)h_1}{2\mu_1+\lambda_1}\partial_1^2 R(x_1) - \frac{2\mu}{\mu_1 h_0}R(x_1) - \frac{\mu h}{\mu_1 h_0}\partial_1^2 u_2^{(0)}(x_1) \\
&= \frac{\rho_1 h_1}{\mu_1}\partial_\tau^2 S(\tau) + \frac{2\mu}{\mu_1 h_0}S(\tau), && (4.4.42) \\
&\frac{1}{3}\partial_1^4 u_2^{(0)}(x_1) + \partial_1^3 R(x_1) = 0, && (4.4.43)
\end{aligned}$$

where

$$R(x_1) = u^{(1)}(x_1) - u^{(2)}(x_1)$$

and

$$S(\tau) = V^1(\tau) - V^2(\tau).$$

From Equation (4.4.42) one can see that both right and left-hand sides must vanish. In addition, it follows from Relationships (4.4.35), (4.4.36) and the fact that the outer layers are assumed to be made with the same thickness and elastic modulus, that one can establish the following algebraic system of equations for A_1 and A_2

$$A_1 + A_2 = 0, \tag{4.4.44}$$

$$q(A_1 - A_2) = 0, \tag{4.4.45}$$

where

$$q = 2\mu b^2 - h_1 h_0 \mu_1 \omega^2. \tag{4.4.46}$$

Given that $A_1, A_2 \neq 0$ and $A_1 \neq A_2$, one can write the following relationship for ω derived from Relationship (4.4.40),

$$\omega = \frac{b}{h_0}\sqrt{\frac{2\mu h_0}{h_1 \mu_1}}. \tag{4.4.47}$$

Again if we constraint the vector $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ to be unit then we obtain (see Relationship (4.4.41))

$$\hat{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}.$$

4.4.6 The static part

The static part in Equations (4.4.42) and (4.4.43) give the solution for $R(x_1)$ and $u_2^{(0)}(x_1)$ in the following closed form

$$R(x_1) = k_1 \exp(r_1 x_1) + k_2 \exp(r_2 x_1), \tag{4.4.48}$$

$$u_2^{(0)} = -3\int R(x_1)dx_1 + L(x_1). \tag{4.4.49}$$

Here $L(x_1)$ is a function given by

$$L(x_1) = \frac{l_1 x_1^3}{6} + \frac{l_2 x_1^2}{2} + l_3 x_1 + l_4. \tag{4.4.50}$$

The quantities k_1, k_2 and k_3 and l_1, l_2, l_3 and l_4 are determined using the boundary conditions along the edges of the layered structure. Also the constants r_1 and r_2 are given by

$$r_1 = \left\{ \frac{3\mu h}{\mu_1 h_0} + \sqrt{\frac{9\mu^2 h^2(2\mu_1+\lambda_1) + 8\mu\mu_1 h_0 h(\mu_1+\lambda_1)}{\mu_1^2 h_0^2(2\mu_1+\lambda_1)}} \right\} \frac{2\mu_1+\lambda_1}{4h(\mu_1+\lambda_1)}$$

and

$$r_2 = \left\{ \frac{3\mu h}{\mu_1 h_0} - \sqrt{\frac{9\mu^2 h^2(2\mu_1+\lambda_1) + 8\mu\mu_1 h_0 h(\mu_1+\lambda_1)}{\mu_1^2 h_0^2(2\mu_1+\lambda_1)}} \right\} \frac{2\mu_1+\lambda_1}{4h(\mu_1+\lambda_1)}.$$

4.4.7 Summary of the chapter

In this chapter a model of wave propagation in a layered structure involving a thin and soft interface has been studied. The asymptotic method has been applied to derive the condition of an imperfect interface across the thin and soft layer.

We have used the asymptotic method to obtain a *lower* dimensional formulation describing propagation of waves associated with the displacement jump across the interface for the anti-plane shear problem (see Section 4.3). For this case, the presence of the lower bound for the frequency ω (see Figure 4.3) is established, corresponding to oscillations in the longitudinal variable x_1 for the displacement jump across the thin layer. We conclude that in Figure 4.3 there is a band gap which characterises the dispersive phenomenon along the three-layered structure. If the frequency of the signal does not exceed the critical value, the oscillation of the structure is accompanied by the displacement jump which decays exponentially. For larger values of the frequency the displacement jump waves propagate along the interface.

In Section 4.4 we have studied the more complicated problem of a plane elasticity deformation for a three-layered structure. We have derived the *lower* dimensional formulation for the leading term of the formal asymptotic expansion of the solution. An example has been given to illustrate the technique that can be used for a more general problem.

Chapter 5

Mathematical models of delamination cracks on imperfect interfaces

In this chapter a mathematical model of a crack along a thin and soft interface layer is studied. Motivation for this work arises from the study of the fracture of ceramic catalytic monolith combustors that are being incorporated into new proto-type designs of gas turbines. This type of soft interface could arise in a ceramic support that has been coated with a layer of high surface area material which contains a dispersed catalyst.

An asymptotic approach is used to replace the interface layer with a set of effective contact conditions. The use of the the phrase *imperfect interface* is to emphasise that the solution (the temperature or displacement field) is allowed to have a non-zero jump across the interface. Compared to classical formulations for cracks in dissimilar media (where ideal contact conditions are specified outside the crack), in our case the gradient field for the temperature or displacement is characterised by a weak logarithmic singularity. The scalar case for the Laplacian operator and the vector elasticity problem are considered. Numerical results are presented for a two-phase elastic strip containing a finite crack on an imperfect interface.

5.1 Motivation and background

Motivation for this chapter arises from the study of the fracture of ceramic catalytic monolith combustors that are being incorporated into new prototype designs of gas turbines. The possibility of crack propagation in the ceramic sup-

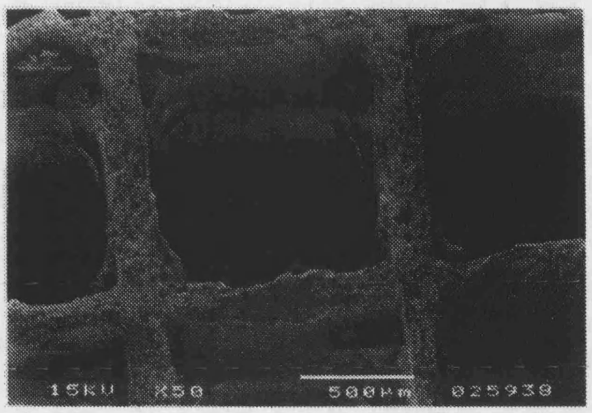

Figure 5.1: Scanning electron micrograph of a face of a catalyst coated monolith (photograph supplied courtesy of University of Bath, UK).

port and methods of calculation have already been described by Antipov *et al.* (1999b). The ceramic monolith consists of an extruded structure that contains a large number of parallel channels, typically comprising 62 cells/cm^2 when each cell is 1.1 mm $\times$1.1 mm square and has 66% of this area exposed. The ceramic surface is coated with a high surface area material (e.g. γ-Al_2O_3) which contains the dispersed catalyst. It is in the catalytic layer (also known as the washcoat), where the combustion reactions take place (eg. $CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$), and the energy associated with this highly exothermic reaction is released. In the gas turbine combustor application, temperatures of the catalyst layer can vary from ambient conditions (when the turbine is not working) up to 1100°C. It is important that the catalyst layer remains firmly bound to the ceramic support structure during this process. If it cracks and shears, then catalyst will be lost from the monolith which would result in a reduced performance of the combustor and potentially lead to damage of components downstream of the combustor. Further information on catalytic combustion and the use of ceramic monolith supports is available in Hayes and Kolaczkowski (1997).

As the surface of the monolith is covered by a layer of catalyst, this gives a two-phase structure. A scanning electron micrograph (SEM) of the face of a catalyst-coated channel is illustrated in Figure 5.1 , taken at a tilt angle of 15° to view the surface of the catalyst layer. Cracks are clearly visible in the layer. The cracks will have occurred as a result of shrinkage of the coated layer (after drying and calcining) or differences in coefficients of thermal expansion as the material is exposed to a wide range of temperatures. The presence of a crack on the surface is not necessarily considered to be a problem. However if the crack propagates and the interface is sheared then this will lead to catalyst loss.

It is documented in engineering literature that the damage of ceramic structures is accompanied by *crack bridging*. In the model presented here we assume that the bridging effect exists along the whole interface surface between the substrate and the layer of catalyst (often, we shall also use the phrases *imperfect interface* or *soft adhesive*) and in addition, a crack with zero tractions on its faces is introduced along the interface contour. We study the problems of heat transfer (or anti-plane shear) and elasticity problems for this two-phase structure.

Mathematical models of interfacial cracks are well-developed in the literature for the cases where ideal contact conditions prevail on an interface surface outside a crack. Plane problems for cracks in dissimilar media were studied by Rice and Sih (1965) and by England (1965). The work of Willis (1971) introduces the integral equation approach for analysis of interfacial cracks including the cases of three dimensions and dynamic cracks. Asymptotic models of elastic adhesive joints were introduced by Klarbring (1991); Klarbring and Movchan (1995, 1998) and Avila-Pozos *et al.* (1999). The adhesive was modelled as a thin and soft layer where effective contact conditions involve continuity of tractions and a linear relation between the traction components and the displacement jump across the adhesive. Laminated structures with linear interfaces were also studied by Bigoni *et al.* (1997).

In this chapter we analyse mathematical models of cracks along imperfect interface boundaries and emphasise on the asymptotic behaviour of the solution and its derivatives near the crack ends and at infinity.

Here, the solution is the so-called outer solution of the asymptotic problem associated with the delamination crack. This solution will be valid outside an infinitesimal neighbourhood of the crack tip.

A possible generalisation of the model would involve a boundary layer near the crack tip and will be described by a solution of the problem posed in an infinite strip containing semi-infinte rectangular void on the left and the three-layer structure on the right. Analysis of a boundary layer in the vicinity of the crack end is not included in this thesis. Here we consider only the analysis of the outer expansions of the solution.

In contrast to the results already published in the literature, on the interface boundary (outside the crack) we allow for a non-zero displacement jump specified as a function of traction components. The presence of this condition affects the asymptotics of the displacement and stress components in the vicinity of the crack ends.

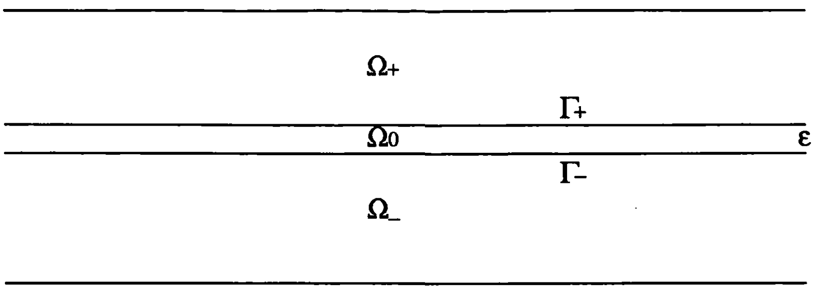

Figure 5.2: The adhesive joint.

5.2 Asymptotic model of adhesive joints

In this section, we discuss the asymptotic model of a thin and soft (adhesive) layer and derive effective interface conditions.

Consider two bodies Ω_+ and Ω_- connected by a thin interface layer Ω_0 of thickness ϵ (see Figure 5.2). Assume that the material occupying Ω_+,Ω_- and Ω_0 is characterised by the shear moduli μ_+, μ_- and $\mu_0 = \epsilon\mu$ respectively, where μ is of the same order as μ_+ and μ_-.

5.2.1 Anti-plane shear

For the case of anti-plane shear, the transverse displacements u^+, u^- and $u^{(0)}$ in Ω_+,Ω_- and Ω_0 satisfy the following equations

$$
\begin{aligned}
\nabla^2 u^{\pm} &= 0 \quad \text{in} \quad \Omega_{\pm}, \\
\nabla^2 u^{(0)} &= 0 \quad \text{in} \quad \Omega_0
\end{aligned}
$$

and

$$
u^+ = u^{(0)}, \quad \mu_+ \frac{\partial u^+}{\partial y} = \mu_0 \frac{\partial u^{(0)}}{\partial y} \quad \text{on} \quad \Gamma_+, \tag{5.2.1}
$$

$$
u^- = u^{(0)}, \quad \mu_- \frac{\partial u^-}{\partial y} = \mu_0 \frac{\partial u^{(0)}}{\partial y} \quad \text{on} \quad \Gamma_-, \tag{5.2.2}
$$

where

$$
\begin{aligned}
\Gamma_+ &= \{(x, y) : y = \epsilon/2\}, \\
\Gamma_- &= \{(x, y) : y = -\epsilon/2\}.
\end{aligned}
$$

Let $t = \epsilon^{-1} y$, so that within Ω_0, $|y| < \epsilon/2$ and hence $|t| < 1/2$. In terms of x

and t

$$\nabla^2 u^{(0)} = \epsilon^{-2}\frac{\partial^2 u^{(0)}}{\partial t^2} + \frac{\partial^2 u^{(0)}}{\partial x^2} = 0.$$

Let $u_0^{(0)}$ denote the leading term of $u^{(0)}$. Then

$$\frac{\partial^2 u_0^{(0)}}{\partial t^2} = 0 \quad \text{in} \quad \Omega_0$$

and therefore

$$u_0^{(0)} = C_1^{(0)}(x) + tC_2^{(0)}(x),$$

with C_1, C_2 being functions of x only. Using the displacement contact conditions in (5.2.1) and (5.2.2) we deduce

$$\begin{aligned} C_1^{(0)} &= \tfrac{1}{2}(u^+(x_1,0) + u^-(x_1,0)), \\ C_2^{(0)} &= u^+(x_1,0) - u^-(x_1,0), \end{aligned}$$

while the traction contact conditions in (5.2.1) and (5.2.2) yield

$$\mu_\pm \frac{\partial u^\pm}{\partial y}(x_1,0) = \mu\frac{\partial u_0^{(0)}}{\partial t} = \mu(u^+(x_1,0) - u^-(x_1,0))$$

to leading order. It is shown that the leading order term of tractions is continuous across the interface layer

$$T(x_1) := \mu_+\frac{\partial u^+}{\partial y}(x_1,0) = \mu_-\frac{\partial u^-}{\partial y}(x_1,0) \tag{5.2.3}$$

and it is proportional to the displacement jump across the interface

$$T(x_1) = \mu(u^+(x_1,0) - u^-(x_1,0)). \tag{5.2.4}$$

In the text below, the quantity μ will be called the stiffness coefficient of the interface layer Ω_0.

5.2.2 Plane strain with isotropic interface layer

For the case of plane strain the materials occupying Ω_0 and $\Omega_\pm$ are characterised by the Lamé elastic moduli $\lambda_0 = \epsilon\lambda$, $\mu_0 = \epsilon\mu$ and $\lambda_\pm$, $\mu_\pm$, where λ, μ have the same order of magnitude as $\lambda_\pm$, $\mu_\pm$.

The displacement vectors $\boldsymbol{u}^\pm$ and $\boldsymbol{u}^{(0)}$ satisfy the Lamé equations of equilibrium

$$L_{\pm}(\boldsymbol{u}^{\pm}) = 0 \quad \text{in} \quad \Omega_{\pm}, \tag{5.2.5}$$

$$L_0(\boldsymbol{u}^{(0)}) = 0 \quad \text{in} \quad \Omega_0, \tag{5.2.6}$$

where

$$L_{\pm}(\boldsymbol{u}) = \mu_{\pm}\nabla^2\boldsymbol{u} + (\lambda_{\pm} + \mu_{\pm})\nabla\nabla\cdot\boldsymbol{u}$$

and

$$\begin{aligned} L_0(\boldsymbol{u}^{(0)}) = {} & \epsilon^{-2}\begin{pmatrix}\mu_0 & 0\\ 0 & \lambda_0+2\mu_0\end{pmatrix}\frac{\partial^2\boldsymbol{u}^{(0)}}{\partial t^2} \\ & + \epsilon^{-1}\begin{pmatrix}0 & \lambda_0+\mu_0\\ \lambda_0+\mu_0 & 0\end{pmatrix}\frac{\partial^2\boldsymbol{u}^{(0)}}{\partial x\partial t} \\ & + \begin{pmatrix}\lambda_0+2\mu_0 & 0\\ 0 & \mu_0\end{pmatrix}\frac{\partial^2\boldsymbol{u}^{(0)}}{\partial x^2}. \end{aligned}$$

The interface contact conditions on $\Gamma_{\pm}$ have the form

$$\boldsymbol{u}^{+} = \boldsymbol{u}^{(0)}, \quad \boldsymbol{\sigma}_{+}^{(2)}(\boldsymbol{u}^{+}) = \boldsymbol{\sigma}^{(2)}(\boldsymbol{u}^{(0)}) \quad \text{on} \quad \Gamma_{+}, \tag{5.2.7}$$

$$\boldsymbol{u}^{-} = \boldsymbol{u}^{(0)}, \quad \boldsymbol{\sigma}_{-}^{(2)}(\boldsymbol{u}^{-}) = \boldsymbol{\sigma}^{(2)}(\boldsymbol{u}^{(0)}) \quad \text{on} \quad \Gamma_{-} \tag{5.2.8}$$

where $\boldsymbol{\sigma}^{(2)} = (\sigma_{12}, \sigma_{22})^T$. It follows from (5.2.6) that the leading term $\boldsymbol{u}_0^{(0)}$ of $\boldsymbol{u}^{(0)}$ is linear with respect to t

$$\boldsymbol{u}_0^{(0)} = \boldsymbol{C}_1^{(0)}(x) + t\boldsymbol{C}_2^{(0)}(x),$$

where the vector functions $\boldsymbol{C}_1^{(0)}, \boldsymbol{C}_2^{(0)}$ are defined from the displacement contact conditions in (5.2.7) and (5.2.8)

$$\boldsymbol{C}_1^{(0)} = \tfrac{1}{2}(\boldsymbol{u}^{+}(x,0) + \boldsymbol{u}^{-}(x,0)), \quad \boldsymbol{C}_2^{(0)} = \boldsymbol{u}^{+}(x,0) - \boldsymbol{u}^{-}(x,0).$$

It follows from the traction conditions in (5.2.7) and (5.2.8) that

$$\begin{aligned} \boldsymbol{\sigma}_{+}^{(2)}(\boldsymbol{u}^{+})|_{(x,0)} = \boldsymbol{\sigma}_{-}^{(2)}(\boldsymbol{u}^{-})|_{(x,0)} &= \frac{1}{\epsilon}\begin{pmatrix}\mu_0 & 0\\ 0 & \lambda_0+2\mu_0\end{pmatrix}\frac{\partial\boldsymbol{u}^{(0)}}{\partial t} \\ &= \begin{pmatrix}\mu & 0\\ 0 & \lambda+2\mu\end{pmatrix}(\boldsymbol{u}^{+}(x,0) - \boldsymbol{u}^{-}(x,0)) \end{aligned} \tag{5.2.9}$$

to leading order.

5.2.3 Plane strain with anisotropic interface layer

Here, we assume that the material in Ω_0 is anisotropic characterised by the constitutive relation

$$\begin{pmatrix} \sigma_{11}^0 \\ \sigma_{22}^0 \\ \sqrt{2}\sigma_{12}^0 \end{pmatrix} = \epsilon \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} \epsilon_{11}^0 \\ \epsilon_{22}^0 \\ \sqrt{2}\epsilon_{12}^0 \end{pmatrix},$$

where $C \equiv (c_{ij})$ is the Hook matrix. The equations of equilibrium in Ω_0 have the form

$$D^T(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})CD(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\boldsymbol{u}^{(0)} = 0 \quad \text{in} \quad \Omega_0,$$

where D is the matrix differential operator specified by

$$D(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial x} \end{pmatrix}.$$

Then the vector of tractions $\boldsymbol{\sigma}_0^{(2)}$ is

$$\begin{aligned} \boldsymbol{\sigma}_0^{(2)}(\boldsymbol{u}^{(0)}) &= D^T(0,1)CD(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\boldsymbol{u}^{(0)} \\ &\sim \begin{pmatrix} \frac{1}{2}c_{33} & \frac{1}{\sqrt{2}}c_{23} \\ \frac{1}{\sqrt{2}}c_{23} & c_{22} \end{pmatrix} \frac{\partial \boldsymbol{u}^{(0)}}{\partial t}. \end{aligned}$$

Similar to the previous (isotropic) case, we show that the leading term of $\boldsymbol{u}^{(0)}$ is linear in t, the tractions $\boldsymbol{\sigma}^{(2)}$ are continuous across the interface and depend linearly on the displacement jump, i.e.

$$\boldsymbol{\sigma}_+^{(2)}(\boldsymbol{u}^+)|_{(x,0)} = \boldsymbol{\sigma}_-^{(2)}(\boldsymbol{u}^-)|_{(x,0)} = \begin{pmatrix} \frac{1}{2}c_{33} & \frac{1}{\sqrt{2}}c_{23} \\ \frac{1}{\sqrt{2}}c_{23} & c_{22} \end{pmatrix} (\boldsymbol{u}^+(x,0) - \boldsymbol{u}^-(x,0)). \tag{5.2.10}$$

The relations (5.2.3)–(5.2.4), (5.2.9) and (5.2.10) give the limit boundary conditions on the interface.

5.3 The Dirichlet problem for a strip with a semi-infinite crack along the imperfect interface

In this section, we present an exact solution of the Dirichlet problem for the Laplacian for a strip containing a semi-infinite crack along the imperfect interface. In terms of physical applications, this model corresponds to the heat transfer problem for a strip with temperature prescribed on the upper and lower parts of the surface, zero flux at the crack boundary and a flux proportional to the temperature difference ahead of the crack. Alternatively, this model can be interpreted as the anti-plane shear problem of elasticity with a thin and soft layer of adhesive placed ahead of the interfacial crack.

The problem is reduced to a scalar Wiener-Hopf functional equation that is solved exactly. The behaviour of the solution at the end of the crack is found by the same technique that was used by Antipov (1993).

We consider the problem of anti-plane shear for the case when the upper and lower sides of the strip are fixed and the crack surface is subject to given tractions. The alternative physical interpretation is related to distribution of temperature in the strip whose exterior surface is kept at a constant temperature, with the heat flux being specified on the crack faces.

5.3.1 Mathematical formulation

Let a three-phase strip contain a semi-infinite delamination crack. Ahead of the crack a thin layer of soft adhesive exists, and following the analysis of Section 5.2, it will be replaced by the discontinuity line where the jump in displacement is proportional to tractions (see Figure 5.3).

Formally, the problem is set as follows

$$\nabla^2 u(x,y) = 0, \quad |x| < \infty, \quad -b < y < 0, \quad 0 < y < a, \tag{5.3.1}$$

$$u(x,a) = u(x,-b) = 0, \qquad |x| < \infty, \tag{5.3.2}$$

$$\mu_+ \frac{\partial u}{\partial y}|_{y=+0} = \mu_- \frac{\partial u}{\partial y}|_{y=-0} = \mu [u]_{y=-0}^{y=+0}, \quad x > 0, \tag{5.3.3}$$

$$\mu_+ \frac{\partial u}{\partial y}|_{y=+0} = \mu_- \frac{\partial u}{\partial y}|_{y=-0} = p(x), \qquad x < 0, \tag{5.3.4}$$

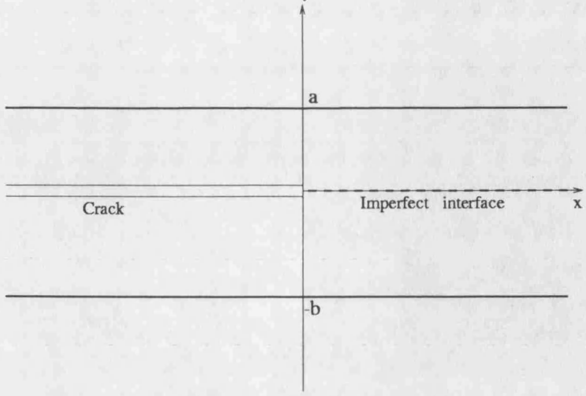

Figure 5.3: Strip with semi-infinite crack along the imperfect interface.

where μ_+,μ_- denote the shear moduli of the elastic material occupying the upper and lower parts of the strip; $p(x)$ characterises the shear load applied to the crack faces; μ is the stiffness coefficient of the interface layer (see Section 5.2). We seek the solution with the finite energy

$$\mathcal{E}(u) = \int_{\mathbb{R}\times[-b,a]} |\nabla u|^2 dxdy. \tag{5.3.5}$$

5.3.2 The auxiliary problem

First, consider an auxiliary problem for the upper part of the strip $(0 < y < a)$

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \qquad |x| < \infty, \quad 0 < y < a, \\ u(x,a) &= 0, \qquad |x| < \infty, \\ \mu_+ \frac{\partial u}{\partial y}|_{y=+0} &= \mathcal{P}(x), \quad |x| < \infty, \end{aligned}$$

where for negative x the function $\mathcal{P}$ is given by $\mathcal{P}(x) = p(x)$, and for $x > 0$ this function is proportional to the displacement jump $\mathcal{P}(x) = \mu\chi(x)$,

$$\chi(x) = u(x,+0) - u(x,-0).$$

Applying the Fourier transform with respect to x

$$u_\alpha(y) = \int_{-\infty}^{+\infty} u(x,y)e^{i\alpha x}dx \tag{5.3.6}$$

we obtain the following boundary value problem for an ordinary differential equation,

$$\begin{aligned} &\left(\tfrac{d^2}{dy^2} - \alpha^2\right)u_\alpha(y) = 0, && 0 < y < a, \\ &u_\alpha(x,a) = 0, && \mu_+ u'_\alpha(x,0) = \mathcal{P}_\alpha. \end{aligned} \tag{5.3.7}$$

The problem (5.3.7) has the following solution

$$u_\alpha(y) = c_1 \cosh\left(\alpha(a-y)\right) + c_2 \sinh\left(\alpha(a-y)\right).$$

Using the boundary conditions in (5.3.7), one can easily notice that the solution is given by

$$u_\alpha(y) = -\frac{\mathcal{P}_\alpha \sinh \alpha(a-y)}{\mu_+ \alpha \cosh(\alpha a)}, \quad 0 < y < a.$$

5.3.3 The displacement jump

In a similar way, we formulate the auxiliary problem for the lower part of the strip ($-b < y < 0$) and find that

$$u_\alpha(y) = \frac{\mathcal{P}_\alpha \sinh \alpha(b+y)}{\mu_- \alpha \cosh(\alpha b)}, \quad -b < y < 0.$$

Thus,

$$\chi_\alpha = u_\alpha(+0) - u_\alpha(-0) = -\frac{\mathcal{P}_\alpha}{\alpha}\left(\frac{\tanh \alpha a}{\mu_+} + \frac{\tanh \alpha b}{\mu_-}\right). \tag{5.3.8}$$

Note that the displacement jump function $\chi(x)$ is unknown everywhere along the real axis.

5.3.4 The exact solution

We introduce a function ψ with supp $\psi(x) \in (-\infty, 0)$ and rewrite the conditions (5.3.3) and (5.3.4) in the form

$$\mu_{\pm}\frac{\partial u}{\partial y}(x, \pm 0) = \mu\chi(x) + \psi(x), \quad |x| < \infty. \tag{5.3.9}$$

We take the Fourier transform of Relationship (5.3.9) with respect to x

$$\mathcal{P}_\alpha = \mu\chi_\alpha + \Phi^-(\alpha), \tag{5.3.10}$$

where

$$\Phi^-(\alpha) = \psi_\alpha = \int_{-\infty}^{0} \psi(\xi)e^{i\alpha\xi}d\xi. \tag{5.3.11}$$

After the function χ_α in Equation (5.3.10) has been replaced by Relationship (5.3.8) we arrive at the Wiener-Hopf functional equation

$$\boxed{\Phi^-(\alpha) = G(\alpha)[\mu\Phi^+(\alpha) + P^-(\alpha)], \quad -\infty < \alpha < +\infty.} \tag{5.3.12}$$

Here,

$$\Phi^+(\alpha) = \int_0^\infty \chi(\xi)e^{i\alpha\xi}d\xi, \quad P^-(\alpha) = \int_{-\infty}^{0} p(\xi)e^{i\alpha\xi}d\xi \tag{5.3.13}$$

and

$$\boxed{G(\alpha) = 1 + \frac{\mu}{\alpha}\left(\frac{\tanh\alpha a}{\mu_+} + \frac{\tanh\alpha b}{\mu_-}\right).} \tag{5.3.14}$$

5.3.5 Properties of the Wiener-Hopf functional equation

It is emphasised that the function $\Phi^+(\alpha)$ is analytic in the upper half-plane $\mathbb{C}^+ = \{\alpha : \text{Im } \alpha > 0\}$ and the functions $P^-(\alpha)$,$\Phi^-(\alpha)$ are analytic in the lower half-plane $\mathbb{C}^- = \{\alpha : \text{Im } \alpha < 0\}$. The boundary values of the unknown functions Φ^+ and Φ^- satisfy the equation (5.3.12) on the real axis.

In the Relationship (5.3.14), the function $G(\alpha)$ is even and has zero increment of arg $G(\alpha)$ along the real axis. It allows for the following factorisation

$$G(\beta) = \frac{X^+(\beta)}{X^-(\beta)}, \quad \beta \in (-\infty, +\infty) \tag{5.3.15}$$

where for real β, $X^\pm(\beta) = X(\beta \pm i0)$ and

$$\begin{aligned} X(\alpha) &= \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln G(\beta) \frac{d\beta}{\beta - \alpha} \right\} \\ &= \exp\left\{ \frac{\alpha}{\pi i} \int_{0}^{+\infty} \ln G(\beta) \frac{d\beta}{\beta^2 - \alpha^2} \right\}, \quad \alpha \in \mathbb{C}\backslash\mathbb{R}^1. \end{aligned} \tag{5.3.16}$$

Note that the function $G(\alpha)$ given by Equation (5.3.14) is bounded at the origin and tends to 1 as $\alpha \to \pm\infty$.

We substitute further (5.3.15) into (5.3.12) and represent $X^+(\alpha)P^-(\alpha)$ as

$$X^+(\alpha)P^-(\alpha) = \Psi^+(\alpha) - \Psi^-(\alpha), \quad \alpha \in \mathbb{R}^1,$$

where $\Psi^\pm(\alpha)$ are the limit values of the function

$$\Psi(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{X^+(\beta)P^-(\beta)}{\beta - \alpha} d\beta. \tag{5.3.17}$$

It yields the following form of Equation (5.3.12)

$$X^-(\alpha)\Phi^-(\alpha) + \Psi^-(\alpha) = \mu X^+(\alpha)\Phi^+(\alpha) + \Psi^+(\alpha) \tag{5.3.18}$$

that is valid in the whole complex plane due to the continuation principle. We are looking for a solution (χ, ψ) integrable in the vicinity of the crack tip. According to Abelian type theorems (see Noble (1958)) the functions $\Phi^\pm(\alpha)$ vanish as $\alpha \to \infty$, $\alpha \in \mathbb{C}^\pm$. Also, $X^\pm(\alpha)$ are bounded and $\Psi^\pm(\alpha)$ vanish as $\alpha \to \infty$, $\alpha \in \mathbb{C}^\pm$. Due to Liouville's theorem (see Section 1.5), the entire function corresponding to (5.3.18) is identically zero. It follows that the functions $\Phi^\pm(\alpha)$ can be represented in the form

$$\boxed{\Phi^-(\alpha) = -\frac{\Psi^-(\alpha)}{X^-(\alpha)}, \quad \alpha \in \mathbb{C}^- \quad , \quad \Phi^+(\alpha) = -\frac{\Psi^+(\alpha)}{\mu X^+(\alpha)}, \quad \alpha \in \mathbb{C}^+.} \tag{5.3.19}$$

5.3.6 Particular example

As an example, consider the case when the right-hand side $p(x)$ in (5.3.4) can be approximated by

$$p(x) \sim \sum_{k=1}^{N} d_k e^{\alpha_k x}, \quad \alpha_k > 0, \quad x < 0 \tag{5.3.20}$$

where d_k, α_k are constant coefficients: $\alpha_1 < \alpha_2 < \ldots < \alpha_N$. Therefore, the function $P^-(\alpha)$ can be written explicitly

$$P^-(\alpha) = \sum_{k=1}^{N} \frac{d_k}{i\alpha + \alpha_k}.$$

After we have evaluated the Cauchy integrals (5.3.17) the solution of the Wiener-Hopf problem when the function $p(x)$ is given by the Relationship (5.3.20) is reduced to the form

$$\Phi^-(\alpha) = -\frac{i}{X^-(\alpha)} \sum_{k=1}^{N} \frac{d_k X_k}{\alpha - i\alpha_k}, \quad \alpha \in \mathbb{C}^-, \tag{5.3.21}$$

$$\Phi^+(\alpha) = \frac{i}{\mu} \sum_{k=1}^{N} \frac{d_k}{\alpha - i\alpha_k}[1 - \frac{X_k}{X^+(\alpha)}], \quad \alpha \in \mathbb{C}^+, \tag{5.3.22}$$

where

$$X_k = X^+(i\alpha_k) = \exp\{\frac{\alpha_k}{\pi} \int_0^\infty \ln G(\beta) \frac{d\beta}{\beta^2 + \alpha_k^2}\},$$
$$Im\ X_k = 0, \quad k = 1, 2, \ldots, N.$$

After applying the inverse Fourier transform

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi^-(\alpha) e^{-i\alpha x} d\alpha, \quad x < 0,$$
$$\chi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi^+(\alpha) e^{-i\alpha x} d\alpha, \quad x > 0. \tag{5.3.23}$$

Using the substitution (5.3.21) and the analytic continuation $X^-(\alpha)$ into the upper half-plane $X^-(\alpha) = (G(\alpha))^{-1} X^+(\alpha)$, $\alpha \in \mathbb{C}^+$, we have

$$\boxed{\psi(x) = \frac{1}{2\pi i}\sum_{k=1}^{N} d_k X_k \int_{-\infty}^{+\infty} \frac{G(\alpha)e^{-i\alpha x}}{X^+(\alpha)(\alpha - i\alpha_k)} d\alpha, \quad x < 0.}$$

We notice that $X^+(\alpha)$ is analytic and does not have zeros in $\mathbb{C}^+$. The application of the residue theorem yields

$$\psi(x) = \sum_{k=1}^{N} d_k \Bigg\{ -\mu X_k \sum_{n=1}^{\infty} \Bigg[\frac{e^{a_n x}}{a\mu_+ a_n (a_n - \alpha_k) X^+(i a_n)} \tag{5.3.24}$$
$$+ \frac{e^{b_n x}}{b\mu_- b_n (b_n - \alpha_k) X^+(i b_n)} \Bigg] + G(i\alpha_k) e^{\alpha_k x} \Bigg\}, \quad x < 0,$$

where $a_k = \frac{\pi}{2a}(2k-1), \;\; b_k = \frac{\pi}{2b}(2k-1)$. Here for the sake of simplicity we have assumed that $\alpha_k \neq a_n$ and $\alpha_k \neq b_n, \;\; k = 1, \dots, N,$ for all $n = 1, 2, \dots, N$.

In particular, as $x \to -\infty$

$$\psi(x) = O(e^{\lambda_0 x}),$$

where $\lambda_0 = \min\{\frac{\pi}{2a}, \frac{\pi}{2b}, \alpha_1\}$.

It follows from (5.3.9) and (5.3.24) that for negative x the function $\chi(x)$ has been determined as well

$$\boxed{\chi(x) = \mu^{-1}\{p(x) - \psi(x)\}.}$$

For positive x, we use the second formula in (5.3.23) and obtain

$$\boxed{\chi(x) = \frac{1}{\mu}\sum_{k=1}^{N} d_k X_k \sum_{n=1}^{\infty} \frac{e^{-\sigma_n x}}{X^-(-i\sigma_n) i G'(-i\sigma_n)(\sigma_n + \alpha_k)}}$$

where $\alpha_k, \sigma_n > 0$, and $-i\sigma_n$ are the elements of the countable set of roots of $G(\alpha)$ in $\mathbb{C}^-$. We note that Im $(X^\pm(\pm i\tau)) = 0, \;\; \tau > 0$ and Im $(iG'(-i\sigma_n)) = 0$.

5.3.7 Asymptotics in a neighbourhood of the crack tip

To obtain the asymptotic behaviour of the functions χ, ψ and their derivatives in the vicinity of the origin we consider first their Fourier transforms $\Phi^\pm(\alpha)$ (see (5.3.21) and (5.3.22)) and analyse these functions as $\alpha \to \infty$. To find the asymptotics of the functions $X^{\pm}$, we represent the integral (5.3.16) in the form

$$\ln X(\alpha) = \frac{\alpha}{\pi i}\left\{\int_0^{\infty} (\ln G(\beta) - \frac{\mu_0}{\beta}\eta(\beta))\frac{d\beta}{\beta^2-\alpha^2} + \mu_0 I(\alpha)\right\},$$

where

$$\mu_0 = \mu(\frac{1}{\mu_+} + \frac{1}{\mu_-}), \quad \eta(\beta) = \begin{cases} 1, & \beta > 1 \\ 0, & \beta < 1 \end{cases}$$

and

$$I(\alpha) = \int_1^{\infty} \frac{d\beta}{\beta(\beta^2-\alpha^2)} = \frac{-1}{2\alpha^2}[\ln(1+\alpha) + \ln(1-\alpha)].$$

The functions $\ln(1\pm\alpha)$ are analytic in the complex plane with the cut which joins the branch points $\alpha = -1, \alpha = 1$ and passes through infinity. The arguments $\Theta_{\pm}$ of $(1\pm\alpha)$ satisfy the condition $-\pi < \Theta_{\pm} < \pi$. As $\alpha \to \infty$ and $\theta = \arg\alpha \in (0,\pi)$, we have $\Theta_+ \to \theta$, $\Theta_- \to \theta - \pi$. If $\theta \in (-\pi, 0)$ and $\alpha \to \infty$ then $\Theta_+ \to \theta$ and $\Theta_- \to \theta + \pi$.

Thus, the integral $I(\alpha)$ exhibits different behaviour at infinity in the upper and lower half-planes

$$I(\alpha) \sim \frac{1}{\alpha^2}(-\ln(\alpha) \pm \frac{\pi i}{2}), \quad \alpha \to \infty, \alpha \in \mathbb{C}^{\pm}, \quad \arg\alpha \in (-\pi,\pi).$$

Now we can establish the asymptotics of the function $\ln X(\alpha)$ at infinity

$$\ln X(\alpha) = -\frac{\mu_0}{\pi i\alpha}\ln\alpha + \frac{c_1^{\pm}}{\pi i\alpha} + O(\alpha^{-3}), \quad \text{as} \quad \alpha\to\infty, \quad \alpha \in \mathbb{C}^{\pm},$$

where

$$c_1^{\pm} = -\int_0^{\infty} (\ln G(\beta) - \frac{\mu_0}{\beta}\eta(\beta))d\beta \pm \frac{\mu_0\pi i}{2}.$$

The asymptotic expression for $X(\alpha)$ takes the form

$$\begin{aligned} X(\alpha) = 1 & -\frac{\mu_0\ln\alpha}{\pi i\alpha} + \frac{c_1^{\pm}}{\pi i\alpha} - \frac{\mu_0^2}{2\pi^2\alpha^2}\ln^2\alpha + c_1^{\pm}\frac{\mu_0}{\pi^2\alpha^2}\ln\alpha \\ & - \frac{(c_1^{\pm})^2}{2\pi^2\alpha^2} + O(\frac{\ln^3\alpha}{\alpha^3}), \quad \text{as}\ \alpha\to\infty, \quad \alpha\in\mathbb{C}^{\pm}. \end{aligned}$$

It follows from (5.3.21) and (5.3.22) that the asymptotics for $\Phi^+(\alpha)$ and $\Phi^-(\alpha)$

are given by

$$\Phi^{\pm}(\alpha) = \sum_{m=0}^{\infty} \alpha^{-(m+1)} \sum_{j=0}^{m} e_{mj}^{\pm} \ln^j \alpha, \quad \alpha \to \infty, \alpha \in \mathbb{C}^{\pm}, \tag{5.3.25}$$

where $e_{mj}^{\pm}$ are constant coefficients. In the text below we shall use several first terms of the expansions (5.3.25), and restrict ourselves to the coefficients $e_{00}^{\pm}, e_{10}^{\pm}$ and $e_{11}^{\pm}$. These quantities are given by

$$e_{00}^{-} = -i \sum_{k=1}^{N} d_k X_k, \quad e_{11}^{-} = \frac{\mu_0}{\pi i} e_{00}^{-},$$

$$e_{10}^{-} = \sum_{k=1}^{N} d_k X_k (\alpha_k + \frac{c_1^{-}}{\pi})$$

and

$$e_{00}^{+} = \frac{i}{\mu} \sum_{k=1}^{N} d_k + \frac{e_{00}^{-}}{\mu}, \quad e_{11}^{+} = \frac{e_{11}^{-}}{\mu},$$

$$e_{10}^{+} = -\frac{1}{\mu} \sum_{k=1}^{N} d_k \alpha_k + \frac{1}{\mu} \sum_{k=1}^{N} d_k X_k (\alpha_k + \frac{c_1^{+}}{\pi}).$$

Hence, as $x \to +0$, the function $\chi(x)$ allows the following asymptotic expansion

$$\boxed{\chi(x) = \Lambda_{00} + \Lambda_{10} x + \Lambda_{11} x \ln x + O(x^2 \ln^2 x),} \tag{5.3.26}$$

where

$$\begin{aligned}
\Lambda_{00} &= \frac{1}{\mu} \sum_{k=1}^{N} d_k (1 - X_k), \\
\Lambda_{10} &= \frac{1}{\mu} \sum_{k=1}^{N} d_k \{\alpha_k + X_k [\frac{1}{\pi} r - \alpha_k + \frac{\mu_0}{\pi}(1-\gamma)]\}, \\
\Lambda_{11} &= -\frac{\mu_0}{\pi \mu} \sum_{k=1}^{N} d_k X_k, \\
r &= \int_0^{\infty} [\ln G(\beta) - \frac{\mu_0}{\beta} \eta(\beta)] d\beta
\end{aligned}$$

and γ is Euler's constant ($\gamma = 0.57721566\ldots$).

Here we have used the following relations between the coefficients Λ_{mj} and e^+_{mj}

$$\Lambda_{00} = -ie^+_{00} \quad , \quad \Lambda_{11} = e^+_{11},$$
$$\Lambda_{10} + \Lambda_{11}(\frac{\pi i}{2} + 1 - \gamma) = -e^+_{10},$$

which can be obtained if we take into account formulae (5.3.23) and

$$I_k(\alpha) = \int_0^\infty e^{i\alpha\tau}\tau^k d\tau = \frac{i\Gamma(k+1)}{\alpha^{k+1}} e^{\frac{\pi i}{2}k},$$

$$L_k(\alpha) = \int_0^\infty e^{i\alpha\tau}\tau^k \ln\tau d\tau = \frac{i\Gamma(k+1)}{\alpha^{k+1}} e^{\frac{\pi i}{2}k}(-\gamma + \sum_{m=1}^{k}\frac{1}{m} + \frac{\pi i}{2} - \ln\alpha),$$
$$0 < \arg\alpha < \pi.$$

The last relation follows from formula 4.352(1) in Gradshteyn and Ryzhik (1980). Alternatively, it can be reduced from the previous relationship in the limit,

$$L_k(\alpha) = \lim_{\beta\to 0}\frac{I_{k+\beta}(\alpha) - I_k(\alpha)}{\beta}.$$

5.3.8 Continuity of the displacement jump

Here we analyse the derivative $d\chi/dx$, as $x \to +0$. It has the logarithmic singularity characterised by the following asymptotics

$$\frac{d}{dx}\chi(x) \sim \Lambda_{11}\ln x + O(1), \quad x \to +0.$$

The case when $x \to -0$ requires the knowledge of function $\psi(x)$ (see (5.3.9)). This function has the asymptotic representation

$$\psi(x) \sim M_{00} - M_{10}x - M_{11}x\ln(-x) + \dots, \quad x \to -0, \tag{5.3.27}$$

obtained in a similar way to the one used for the function χ when $x \to +0$. Here, the coefficients M_{00}, M_{10} and M_{11} are defined by

$$M_{00} = \sum_{k=1}^{N} d_k X_k, \qquad M_{11} = -\frac{\mu_0}{\pi} M_{00},$$
$$M_{10} = \sum_{k=1}^{N} d_k X_k [\frac{r}{\pi} - \alpha_k + \frac{\mu_0}{\pi}(1-\gamma)].$$

It follows from (5.3.9) that

$$\chi(-0) = \frac{1}{\mu}(p(-0) - \psi(-0)) = \frac{1}{\mu}\sum_{k=1}^{N} d_k(1 - X_k) = \chi(+0)$$

and therefore, the displacement jump $[u](x)$ is continuous at the crack tip ($x = 0$). Moreover, in the neighbourhood of the point $x = 0$ the function $\chi(x)$ allows the following expansion

$$\chi(x) = \Lambda_{00} + \Lambda_{10}x + \Lambda_{11}x \ln|x| + \dots, \quad x \to 0. \tag{5.3.28}$$

For the derivative $\psi'(x)$, the expression (5.3.27) yields

$$\boxed{\psi'(x) = -M_{11}\ln(-x) + O(1), \quad x \to -0.}$$

From (5.3.28) we deduce

$$\boxed{\chi'(x) = \Lambda_{11}\ln|x| + O(1), \quad x \to 0.} \tag{5.3.29}$$

Since the displacement jump $\chi(x)$ is continuous at $x = 0$, and the function $\psi(x)$ is discontinuous,

$$\psi(+0) = 0, \quad \psi(-0) = M_{00} \neq 0,$$

we conclude (see (5.3.9)) that the traction

$$\sigma_{zy}(x,0) = \mu_+ \frac{\partial u}{\partial y}(x,+0) = \mu_- \frac{\partial u}{\partial y}(x,-0)$$

is bounded but discontinuous at the crack tip,

$$[\sigma_{zy}(x,0)]_{x=-0}^{x=+0} = -\psi(-0) = -M_{00}.$$

The Relationship (5.3.29) shows that the stress component $\sigma_{zx}(x,0)$ has the logarithmic singularity at $x = 0$.

5.4 The Neumann problem for a strip with a semi-infinite crack along the imperfect interface

In this section, we study the Neumann boundary value problem for the domain of the same configuration as in the previous section. The qualitative structure of the asymptotics in the vicinity of the crack tip does not change. However, the behaviour of the solution at infinity is different from the case when the temperature values are specified on the upper and lower parts of the boundary of the strip. The exact solution is found by the factorisation method. Explicit asymptotic formulae for the temperature jump are obtained when $x \to \pm\infty$. It is shown that the temperature jump decays exponentially at infinity along the interface outside the crack and is bounded at infinity along the crack. The formulation is similar to Section 5.3, with Dirichlet boundary conditions on the upper and lower parts of the strip being replaced by the homogeneous Neumann data (see Figure 5.3). The unknown function $u(x, y)$ satisfies the equation (5.3.1) and the contact conditions (5.3.3) and (5.3.4). Instead of (5.3.2), we assume that

$$\frac{\partial u}{\partial y}(x, a) = \frac{\partial u}{\partial y}(x, -b) = 0, \quad |x| < \infty.$$

We seek the solution in the class of functions with the finite energy integral (5.3.5) and with the following behaviour at infinity

$$|u(x, y)| < C_1, \quad \text{as } x \to -\infty$$

and

$$|u(x, y)| < C_2 e^{-\delta x}, \quad \text{as } x \to +\infty,$$

uniformly with respect to $-b < y < a$, where C_1, C_2 and δ are positive constants. To specify the solution uniquely, we impose the following orthogonality condition

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial y}(x, \pm 0)dx = 0. \tag{5.4.1}$$

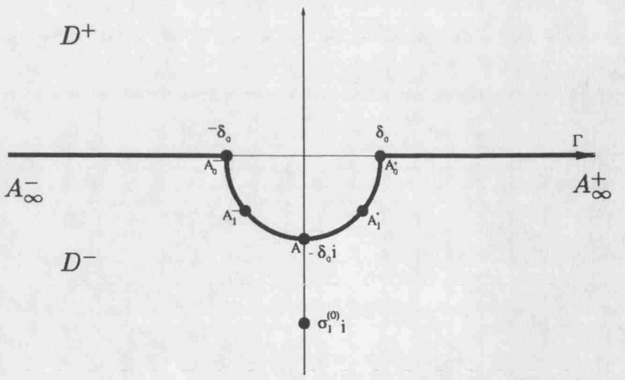

Figure 5.4: The contour Γ.

5.4.1 Analysis of the Wiener-Hopf functional equation

The choice of the above function class ensures the existence of the Fourier transform $u_\alpha(y)$ defined by (5.3.6) in the strip $-\delta < \text{Im } \alpha < 0$.

Following the same pattern as in Section 5.3 we consider two auxiliary formulations in the upper and the lower parts of the strip and obtain the following functional equation of the Wiener-Hopf type

$$\Phi^-(\alpha) = G(\alpha)[\mu\Phi^+(\alpha) + P^-(\alpha)], \tag{5.4.2}$$
$$\alpha \in \Gamma, \quad \Gamma = \{\alpha : \text{Im } \alpha = -\delta_0 \in (-\delta, 0)\},$$

where

$$\boxed{G(\alpha) = 1 + \frac{\mu}{\alpha}\left(\frac{\coth \alpha a}{\mu_+} + \frac{\coth \alpha b}{\mu_-}\right).} \tag{5.4.3}$$

Here the constants μ, μ_+ and μ_- are the same as in Section 5.3. The functions $\Phi^\pm(\alpha)$ and $P^-(\alpha)$ are defined by (5.3.11),(5.3.13): $\Phi^+(\alpha)$ is analytic in the domain D^+, and $\Phi^-(\alpha), P^-(\alpha)$ are analytic in D^-, where

$$D^+ = \{\alpha : \text{Im } \alpha > -\delta_0\}, \quad D^- = \{\alpha : \text{Im } \alpha < -\delta_0\}.$$

Here, the quantity $\delta_0 > 0$ has been chosen in such a way that the strip $\{-\delta_0 < \text{Im } \alpha < 0\}$ does not include any roots of the function $G(\alpha)$.

5.4.2 Properties of the functional equation

The function $G(\alpha)$ has the second-order pole at the point $\alpha = 0 \in D^+$. In order to compute the increment of $\Theta = \arg G(\alpha)$, we deform the contour Γ to the shape shown in Figure 5.4. The positive direction of contour Γ is chosen in such a way that the domain D^+ is on the left. We consider the following points on the deformed contour

$$\begin{aligned} A_\infty^\pm : &\quad \alpha = \pm\infty - i0, \\ A_0^\pm : &\quad \alpha = \pm\delta_0 - i0, \\ A_1^\pm : &\quad \alpha = \frac{\delta_0}{\sqrt{2}}(\pm 1 - i), \\ A : &\quad \alpha = -\delta_0 i. \end{aligned}$$

At the starting point A_∞^-, the function $G(\alpha) = 1$. At this point, we set $\Theta = 2\pi$. As α runs from A_∞^- to A_0^-, the argument Θ of $G(\alpha)$ does not change. As the radius δ_0 of the circle in Figure 5.4 tends to zero, the value $G(A_0^-)$ behaves as $G(A_0^-) \sim \delta^{-2}\mu_1$, where $\mu_1 = \mu(\frac{1}{a\mu_+} + \frac{1}{b\mu_-})$. As α travels from A_0^- to A_1^-, the argument of the principal part of G changes from 2π to $\frac{3\pi}{2}$, and for small δ_0, the real and imaginary parts of $G(A_1^-)$ have the following leading terms

$$\operatorname{Re} G(A_1^-) \sim 1, \quad \operatorname{Im}\ G(A_1^-) \sim -\frac{\mu_1 i}{\delta_0^2}.$$

Next, the argument Θ reduces further down to π at the point A, and $G(A) \sim -\frac{\mu_1}{\delta_0^2}$ as $\delta_0 \to 0$. In a similar way it can be shown that, as $\delta_0 \to 0$,

$$\begin{aligned} \operatorname{Re} G(A_1^+) &\sim 1, \quad Im\ G(A_1^+) \sim \delta_0^{-2}\mu_1, \quad \Theta|_{\alpha=A_1^+} \sim \frac{\pi}{2}, \\ G(A_0^+) &\sim \delta_0^{-2}\mu_1, \quad \Theta|_{\alpha=A_0^+} = 0, \\ G(A_\infty^+) &= 1, \quad \Theta|_{\alpha=A_\infty^+} = 0. \end{aligned}$$

Thus, we have shown that $\arg G(\alpha)$ changes from 2π down to 0 as α travels from A_∞^- to A_∞^+ along the contour Γ (see Figure 5.5). According to the definition of the index of the Riemann boundary value problem (see Gakhov (1966)), it is equal to

$$\kappa = -\mathrm{ind}_\Gamma G(\alpha) = -\frac{1}{2\pi}[\arg G(\alpha)]_\Gamma = 1.$$

It shows that the solution of the Riemann boundary value problem (5.4.2) will

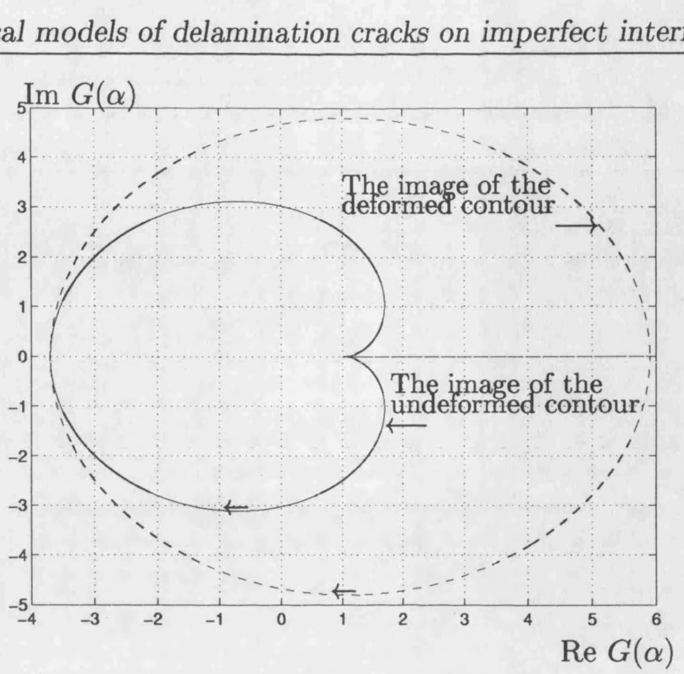

Figure 5.5: The parametrically defined function $G(\alpha)$, with α travelling along the contour Γ.

be specified up to an arbitrary constant. We take into account that

$$\left[\arg \frac{\alpha - i}{\alpha + i}\right]_\Gamma = 2\pi,$$

and introduce the quantity $\frac{\alpha-i}{\alpha+1}G(\alpha)$ which satisfies the condition

$$\left[\arg \left(\frac{\alpha - i}{\alpha + i} G(\alpha)\right)\right]_\Gamma = 0.$$

The last condition can be factorised as follows

$$\frac{t-i}{t+i}G(t) = \frac{X^+(t)}{X^-(t)}, \quad t \in \Gamma, \tag{5.4.4}$$

where

$$X^\pm(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \ln\left(\frac{t-i}{t+i}G(t)\right)\frac{dt}{t-\alpha}\right\}, \quad \alpha \in D^\pm.$$

It follows from (5.4.2),(5.4.4) that

$$\Phi^-(t)(t-i)X^-(t) + \Psi^-(t) = \mu\Phi^+(t)(t+i)X^+(t) + \Psi^+(t), \quad t \in \Gamma, \tag{5.4.5}$$

where $\Psi^{\pm}(t)$ are the one-sided limits of the function

$$\Psi(\alpha) = \frac{1}{2\pi i}\int_{\Gamma}\frac{(t+i)X^{+}(t)P^{-}(t)}{t-\alpha}dt, \quad \alpha \in \mathbb{C}\backslash\Gamma.$$

5.4.3 Comparison with the Dirichlet problem

Compared to equation (5.3.18) for the Dirichlet problem, in (5.4.5) we have additional factors $t \mp i$ in front of $X^{\pm}$. Consequently the entire function is bounded at infinity and is equal to a constant C everywhere in the complex plane. Thus, the functions $\Phi^{\pm}$ are given by

$$\boxed{\Phi^{+}(\alpha) = \frac{C-\Psi^{+}(\alpha)}{\mu(\alpha+i)X^{+}(\alpha)}, \quad \alpha \in D^{+},}$$

and

$$\boxed{\Phi^{-}(\alpha) = \frac{C-\Psi^{-}(\alpha)}{(\alpha-i)X^{-}(\alpha)}, \quad \alpha \in D^{-}.}$$

The constant C is determined by the condition (5.4.1) that can be rewritten in the form

$$\mu\Phi^{+}(0) + P^{-}(0) = 0,$$

where $P^{-}(0) = \int_{-\infty}^{0} p(x)dx$ is finite since the external load $p(x)$ is integrable on $(-\infty, 0)$. We have

$$C = \Psi^{+}(0) - iX^{+}(0)P^{-}(0).$$

5.4.4 Special case

As in the previous section, we consider the particular case when the applied load is represented by (5.3.20). Then the solution of the Wiener-Hopf problem (5.4.2) is given by

$$\begin{aligned}
\Phi^{+}(\alpha) &= \frac{i}{\mu}\sum_{k=1}^{N}\frac{1}{\alpha - i\alpha_k}\left\{d_k - \frac{\alpha d_k^0}{(\alpha+i)X^{+}(\alpha)\alpha_k}\right\}, \\
\Phi^{-}(\alpha) &= \frac{-i\alpha}{(\alpha-i)X^{-}(\alpha)}\sum_{k=1}^{N}\frac{d_k^0}{\alpha_k(\alpha - i\alpha_k)},
\end{aligned} \tag{5.4.6}$$

where $d_k^0 = d_k(\alpha_k + 1)X_k$, $X_k = X^{+}(i\alpha_k)$. Analysing asymptotics of $\Phi^{\pm}(\alpha)$ as $\alpha \to \infty$, $\alpha \in \mathbb{C}^{\pm}$ we obtain the representation of the function $\chi(x)$ as $x \to 0$. The asymptotic formula is the same as in the previous section (see (5.3.28)) with the different coefficients $\Lambda_{00}, \Lambda_{10}, \Lambda_{11}$. We confine ourselves to give the principal coefficient

$$\Lambda_{00} = \frac{1}{\mu} \sum_{k=1}^{N} d_k \left(1 - X_k \frac{\alpha_k + 1}{\alpha_k} \right).$$

As in the previous section, it is observed that the shear stress $\sigma_{zx}(x, 0)$ has the logarithmic singularity at $x = 0$ and $\sigma_{zy}(x, 0)$ possesses a discontinuity of the first kind.

5.4.5 The displacement jump

The functions $\psi(x)$ and $\chi(x)$ are determined as the inverse Fourier transforms

$$\psi(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_\Gamma \Phi^-(\alpha) e^{-i\alpha x} d\alpha \ , \ x < 0 \\ 0 \qquad\qquad\qquad\quad , \ x > 0 \end{cases} \tag{5.4.7}$$

and

$$\chi(x) = \begin{cases} -\dfrac{1}{\mu}(\psi(x) - p(x)) \quad , \ x < 0 \\ \dfrac{1}{2\pi} \displaystyle\int_\Gamma \Phi^+(\alpha) e^{-i\alpha x} d\alpha \ , \ x > 0. \end{cases}$$

Substituting Φ^- from (5.4.6) into (5.4.7) we continue analytically $X^-(\alpha)$ into D^+ (see (5.4.4)) and observe that the integrand has a simple pole at the point $\alpha = 0$. Using the residue theorem we obtain

$$\psi(x) = A + \sum_{k=1}^{N} d_k G(i\alpha_k) e^{\alpha_k x} \tag{5.4.8}$$

$$- \sum_{n=1}^{\infty} \frac{\mu}{\pi n} \left(\frac{1}{\mu_+} \mathcal{F}(a_n^{(0)}, x) + \frac{1}{\mu_-} \mathcal{F}(b_n^{(0)}, x) \right), \quad x < 0,$$

where

$$A = \frac{\mu_1}{X^+(0)} \sum_{k=1}^{N} \frac{d_k^0}{\alpha_k^2}$$

and

$$\mathcal{F}(c_n, x) = \frac{e^{c_n x}}{(c_n + 1) X^+(ic_n)} \sum_{k=1}^{N} \frac{c_n d_k^0}{(c_n - \alpha_k)\alpha_k}.$$

The positive coefficients α_k are the same as in the previous section (see formula (5.3.20)), and the constants $a_n^{(0)}, b_n^{(0)}$ are given by

$$a_n^{(0)} = \frac{\pi n}{a}, \quad b_n^{(0)} = \frac{\pi n}{b}.$$

In contrast to the Dirichlet problem studied in the previous section, the functions $\psi(x)$ and $\chi(x)$ do not vanish as $x \to -\infty$:

$$\psi(x) = A + O(e^{\lambda_1 x}),$$

$$\chi(x) = -\frac{1}{\mu}A + O(e^{\lambda_1 x}),$$

where $\lambda_1 = \min\{\alpha_1, \frac{\pi}{a}, \frac{\pi}{b}\}$.

For positive x, the function ψ vanishes (see (5.4.7)), and χ is determined by

$$\boxed{\chi(x) = -\frac{i}{\mu}\sum_{n=1}^{\infty} \frac{e^{-\sigma_n^{(0)}x}}{X^-(-i\sigma_n^{(0)})(\sigma_n^{(0)}+1)G'(-i\sigma_n^{(0)})} \sum_{k=1}^{N} \frac{d_k^0 \sigma_n^{(0)}}{(\sigma_n^{(0)}+\alpha_k)\alpha_k}.}$$

Here $-i\sigma_n^{(0)}$ denotes the roots of the function $G(\alpha)$ (see (5.4.3)) in D^-. As $x \to +\infty$, $\chi(x)$ vanishes exponentially:

$$\chi(x) = O(e^{-\sigma_1^{(0)}x}), \quad x \to +\infty.$$

We note that $\sigma_1^{(0)}$ determines the range of change of radius of the semicircle on the contour Γ (see Figure 5.4): $0 < \delta_0 < \sigma_1^{(0)}$.

5.5 Infinite plane with a semi-infinite delamination crack

In this section, we analyse a solution of the model problem in a semi-infinite crack in a two-phase plane with the phases being separated by a line of imperfect interface. In contrast with previous sections, it is shown that the solution is characterised by algebraic asymptotics at infinity. We consider an infinite two-phase plane with the line of imperfect interface along the x-axis. A semi-infinite delamination crack is introduced on $(x,y): y=0,\ x<0$ (see Figure 5.3). The problem of anti-plane shear is formulated as follows. The displacement function

u satisfies the Laplace equation

$$\nabla^2 u(x,y) = 0, \quad \text{in } \mathbb{R}^2 \backslash \{y=0\},$$

the contact conditions

$$\mu_+ \frac{\partial u}{\partial y}|_{y=+0} = \mu_- \frac{\partial u}{\partial y}|_{y=-0},$$
$$\mu_\pm \frac{\partial u}{\partial y}|_{y=\pm 0} = \mu[u], \quad \text{for } x > 0$$

and the following boundary condition on the crack faces

$$\mu_\pm \frac{\partial u}{\partial y}|_{y=\pm 0} = p(x), \quad x < 0.$$

Here, the right-hand side $p(x)$ is chosen in such a way that the problem has the solution with the finite energy integral (5.3.5).

Taking the Fourier transforms with respect to x (see (5.3.6)) and following the same pattern as in Section 5.3 we obtain the functional equation of the Wiener-Hopf type on the real axis:

$$\Phi^-(\alpha) = G(\alpha)(\mu\Phi^+(\alpha) + P^-(\alpha)), \quad \alpha \in (-\infty, +\infty), \tag{5.5.1}$$

where

$$G(\alpha) = 1 + \frac{\mu_0}{|\alpha|}.$$

We note that the function G is real and positive on the real axis, and can be factorised in the form

$$G(\alpha) = \frac{X^+(\alpha)}{X^-(\alpha)},$$

where the functions $X^\pm(\alpha)$ are the limiting values of

$$X(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \ln\,(1 + \frac{\mu_0}{|\beta|})\frac{d\beta}{\beta - \alpha}\right\}. \tag{5.5.2}$$

As in Section 5.3, the relationship (5.5.1) is reduced to the form (5.3.18), where the functions $X^\pm$ are defined by (5.5.2). The solution of (5.5.1) is given by (5.3.19). To obtain the behaviour of the displacement in a neighbourhood of the crack tip and at infinity, we analyse the solution $\Phi^+(\alpha), \Phi^-(\alpha)$ as $\alpha \to \infty$ or $\alpha \to 0$.

In the vicinity of the crack tip, the behaviour of the displacement is similar to the one discussed in Sections 5.3 and 5.4 (see (5.3.26)). The qualitative difference between these two cases is observed when $x \to -\infty$. In contrast with Sections 5.3 and 5.4, the functions $X^{\pm}$ are characterised by the following asymptotics

$$X^{+}(\alpha) = (-i\alpha)^{-\frac{1}{2}}(\mu_0^{\frac{1}{2}} + o(1)) \quad \text{as } \alpha \to 0, \quad \alpha \in \mathbb{C}^{+},$$
$$X^{-}(\alpha) = (i\alpha)^{\frac{1}{2}}(\mu_0^{-\frac{1}{2}} + o(1)) \quad \text{as } \alpha \to 0, \quad \alpha \in \mathbb{C}^{-}.$$

5.5.1 Special case

As in the previous two sections, we consider the special case of applied load specified by (5.3.20). Then the formulae (5.3.21), (5.3.22) can be used to determine the asymptotics of functions $\Phi^{\pm}(\alpha)$ at the point $\alpha = 0$:

$$\begin{aligned} \Phi^{+}(\alpha) &\sim B_{+}, \quad \alpha \to 0, \quad \alpha \in \mathbb{C}^{+}, \\ \Phi^{-}(\alpha) &\sim (i\alpha)^{-\frac{1}{2}} B_{-}, \quad \alpha \to 0, \quad \alpha \in \mathbb{C}^{-}, \end{aligned} \tag{5.5.3}$$

where

$$B_{+} = -\frac{1}{\mu}\sum_{k=1}^{N}\frac{d_k}{\alpha_k} \quad \text{and } B_{-} = \mu_0^{\frac{1}{2}}\sum_{k=1}^{N}\frac{d_k}{\alpha_k}X^{+}(i\alpha_k).$$

Using the definition (5.3.11) and the second asymptotic formula (5.5.3) we obtain the asymptotic behaviour of $\psi(\xi)$ as $x \to -\infty$

$$\psi(\xi) \sim \frac{B_{-}}{\sqrt{\pi}}(-x)^{-\frac{1}{2}}. \tag{5.5.4}$$

To derive (5.5.4), we have used the Abelian type theorem (see Section 1.5). Thus, due to (5.3.9), the function χ characterising the displacement jump has the square root asymptotics at infinity

$$\boxed{\chi(x) \sim -\frac{B_{-}}{\mu\sqrt{\pi}}(-x)^{-\frac{1}{2}}, \quad x \to -\infty.}$$

As $x \to +\infty$, it can be seen from (5.3.22) that the function $\chi(x)$ decays as follows

$$\chi(x) = O(x^{-1}), \quad x \to +\infty.$$

We note that the function χ, specified in the previous sections for a finite strip, decays exponentially as $x \to +\infty$.

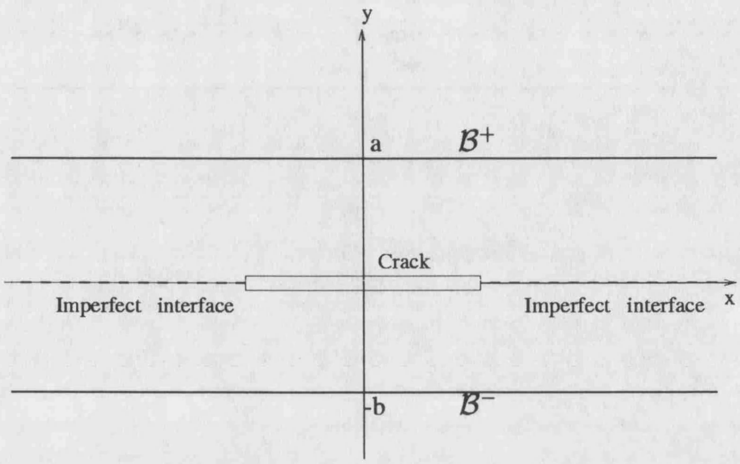

Figure 5.6: Strip with finite crack along the imperfect interface.

5.6 Plane strain problem for a strip with a finite crack along an imperfect interface

In this section we study a plane strain problem for a two-phase elastic strip with a finite crack along an imperfect interface. The problem is formulated in terms of a system of singular integral equations. We consider the domain with the same geometry as in Section 5.3 but with the crack along the imperfect interface being finite (see Figure 5.6). Let the displacement vector $\boldsymbol{u} = (u, v)$ satisfy the homogeneous equilibrium equations

$$\nabla^2 \boldsymbol{u} + \frac{1}{1 - 2\nu_\pm} \nabla\nabla \cdot \boldsymbol{u} = 0 \quad \text{in} \quad \mathcal{H},$$

where $\mathcal{H} = \{(x, y) : x \in \mathbb{R}, \ \ y \in (-b, a) \backslash \{0\}\}$ and the subscript "+" is allocated for the upper layer $0 < y < a$, and the subscript "−" corresponds to the lower layer $-b < y < 0$; $\nu_\pm$ denotes the Poisson ratio of the material of the upper and lower layers. Homogeneous traction conditions are posed on the upper and lower parts of the boundary

$$\tau_{xy} = \sigma_y = 0 \quad \text{on } \mathcal{B}^+ \text{ and } \mathcal{B}^-,$$

where

$$\mathcal{B}^+ = \{(x, y) : x \in \mathbb{R}, \ \ y = a\}$$

and

$$\mathcal{B}^- = \{(x,y) : x \in \mathbb{R},\ \ y = -b\}.$$

The traction conditions are specified on the crack faces

$$\tau_{xy} = f_1(x), \quad \sigma_y = f_2(x) \ \text{ as } |x| < c, \ \ y = 0, \tag{5.6.1}$$

and the interface conditions are given outside the crack by

$$\begin{aligned} \tau_{xy}(x,0) &= \alpha_{11}[u] + \alpha_{12}[v], \\ \sigma_y(x,0) &= \alpha_{12}[u] + \alpha_{22}[v], \\ [\sigma_y] = [\tau_{xy}] &= 0 \ \text{ as } |x| > c, \ \ y = 0. \end{aligned} \tag{5.6.2}$$

Here

$$[g] = g|_{y=+0} - g|_{y=-0}.$$

We seek the solution with finite elastic energy, and assume that the displacement field decays at infinity.

5.6.1 The formulation in terms of the Airy stress function

We introduce the following notations for displacement jump components

$$\chi_1(x) = [u](x), \quad \chi_2(x) = [v](x), \quad |x| < \infty \tag{5.6.3}$$

and for traction components

$$\sigma(x) = \sigma_y(x,0), \quad \tau(x) = \tau_{xy}(x,0), \quad |x| < \infty.$$

Let U denote the Airy stress function. Then the stress components and derivatives of displacements are given by

$$\sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_x = \frac{\partial^2 U}{\partial y^2} \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}$$

and

$$2\mu_{\pm}\frac{\partial u}{\partial x} = (1-\nu_{\pm})\frac{\partial^2 U}{\partial y^2} - \nu_{\pm}\frac{\partial^2 U}{\partial x^2},$$
$$2\mu_{\pm}\frac{\partial v}{\partial y} = (1-\nu_{\pm})\frac{\partial^2 U}{\partial x^2} - \nu_{\pm}\frac{\partial^2 U}{\partial y^2}, \tag{5.6.4}$$
$$\mu_{\pm}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = -\frac{\partial^2 U}{\partial x \partial y},$$

where $\mu_{\pm}$ denotes the shear moduli of the material in the upper and lower parts of the composite strip.

5.6.2 Auxiliary problem

First, we consider the upper part of the strip ($0 < y < a$). The function U satisfies the boundary value problem

$$\Delta^2 U(x,y) = 0 \quad |x| < \infty, \quad 0 < y < a,$$
$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial x \partial y} = 0, \quad |x| < \infty, \quad y = a,$$
$$\frac{\partial^2 U}{\partial x^2} = \sigma(x), \quad -\frac{\partial^2 U}{\partial x \partial y} = \tau(x), \quad |x| < \infty, \quad y = 0.$$

Taking the Fourier transform with respect to x

$$U_\alpha(y) = \int_{-\infty}^{+\infty} U(x,y)e^{i\alpha x}dx,$$

we obtain

$$U_\alpha^{(IV)}(y) - 2\alpha^2 U_\alpha''(y) + \alpha^4 U_\alpha(y) = 0, \; 0 < y < a, \tag{5.6.5}$$
$$U_\alpha(a) = U_\alpha'(a) = 0, \tag{5.6.6}$$
$$-\alpha^2 U_\alpha(0) = \sigma_\alpha, \quad i\alpha U_\alpha'(0) = \tau_\alpha, \tag{5.6.7}$$

where $\sigma_\alpha, \tau_\alpha$ denote the Fourier transforms of tractions. The solution of the system (5.6.5)–(5.6.7) has the form

$$U_\alpha(y) = C_1 \cosh(a-y)\alpha + C_2 \sinh(a-y)\alpha$$
$$+C_3(a-y)\cosh(a-y)\alpha + C_4(a-y)\sinh(a-y)\alpha, \tag{5.6.8}$$

where the constants C_1, C_2, C_3, C_4 are given by

$$\begin{aligned}
C_1 &= 0,\\
C_2 &= \frac{a\sinh(\alpha a)}{i\alpha d(\alpha a)}\tau_\alpha - \frac{\sinh(\alpha a) + \alpha a\cosh(\alpha a)}{\alpha^2 d(\alpha a)}\sigma_\alpha,\\
C_3 &= -\alpha C_2,\\
C_4 &= -\frac{a\sinh(\alpha a)}{d(\alpha a)}\sigma_\alpha - \frac{\sinh(\alpha a) - \alpha a\cosh(\alpha a)}{i\alpha d(\alpha a)}\tau_\alpha
\end{aligned}$$

and $d(\beta) = \sinh^2\beta - \beta^2$.

We need the displacement components on the upper boundary of the interface. The Relationships (5.6.4) and (5.6.8) yield the following expressions for the Fourier transforms

$$\begin{aligned}
-2i\alpha\mu_+ u_\alpha(+0) &= \frac{(1-2\nu_+)\sinh^2(\alpha a) + \alpha^2 a^2}{d(\alpha a)}\sigma_\alpha\\
&+\frac{2(1-\nu_+)\tau_\alpha}{id(\alpha a)}(\alpha a - \sinh(\alpha a)\cosh(\alpha a)),\\
2\alpha\mu_+ v_\alpha(+0) &= \frac{(1-2\nu_+)\sinh^2(\alpha a) + \alpha^2 a^2}{id(\alpha a)}\tau_\alpha\\
&-\frac{2(1-\nu_+)\sigma_\alpha}{d(\alpha a)}(\alpha a + \sinh(\alpha a)\cosh(\alpha a)).
\end{aligned} \tag{5.6.9}$$

In general, one can add arbitrary rigid-body translations and rotations. Here, we have assumed that these terms are equal to zero, which is consistent with the assumption of decay at infinity introduced at the beginning of the section.

In a similar way, one can write the boundary value problem for U_α in the lower part of the strip and obtain the Fourier transforms of the displacement components $u_\alpha(-0)$ and $v_\alpha(-0)$ which have the form (5.6.9) where a should be replaced by $-b$ and μ_+, ν_+ should be replaced by μ_-, ν_-. Thus, the Fourier transforms of the functions χ_1, χ_2 (see (5.6.3)) are

$$\begin{aligned}
i\alpha\chi_{1\alpha} &= -g(\alpha)\sigma_\alpha + ih_-(\alpha)\tau_\alpha,\\
i\alpha\chi_{2\alpha} &= -ih_+(\alpha)\sigma_\alpha + g(\alpha)\tau_\alpha,
\end{aligned} \tag{5.6.10}$$

where

$$g(\alpha) = \frac{\kappa_1^+ \sinh^2(\alpha a) + \kappa_0^+ \alpha^2 a^2}{d(\alpha a)} - \frac{\kappa_1^- \sinh^2(\alpha b) + \kappa_0^- \alpha^2 b^2}{d(\alpha b)}, \tag{5.6.11}$$

$$h_\pm(\alpha) = \frac{\kappa_2^+ (2\alpha a \pm \sinh(2\alpha a))}{d(\alpha a)} + \frac{\kappa_2^- (2\alpha b \pm \sinh(2\alpha b))}{d(\alpha b)}, \tag{5.6.12}$$

$$\kappa_0^\pm = \frac{1}{2\mu_\pm}, \quad \kappa_1^\pm = \frac{1-2\nu_\pm}{2\mu_\pm}, \quad \kappa_2^\pm = \frac{1-\nu_\pm}{2\mu_\pm}.$$

5.6.3 The system of integral equations

Next, we consider the interface conditions (5.6.2). We introduce some unknown functions Ψ_1, Ψ_2 in such a way that

$$\begin{aligned} \tau_{xy}(x,0) &= \alpha_{11}[u](x) + \alpha_{12}[v](x) + \Psi_1(x), \\ \sigma_y(x,0) &= \alpha_{12}[u](x) + \alpha_{22}[v](x) + \Psi_2(x), \quad |x| < \infty, \end{aligned} \tag{5.6.13}$$

and note that supp $\Psi_j(x) \subset (-c,c)$, $j = 1,\ 2$. In terms of the Fourier transforms, equations (5.6.13) can be written as

$$\begin{aligned} \tau_\alpha &= \alpha_{11}\chi_{1\alpha} + \alpha_{12}\chi_{2\alpha} + \Psi_{1\alpha}, \\ \sigma_\alpha &= \alpha_{12}\chi_{1\alpha} + \alpha_{22}\chi_{2\alpha} + \Psi_{2\alpha}, \end{aligned} \tag{5.6.14}$$

where

$$\Psi_{j\alpha} = \int_{-c}^{c} \Psi_j(\xi) e^{i\alpha\xi} d\xi, \quad j = 1, 2. \tag{5.6.15}$$

Therefore, we solve for $\chi_{1\alpha}$, $\chi_{2\alpha}$ in (5.6.14) using the Relationships (5.6.10),

$$\begin{aligned} \sigma_\alpha = \frac{-i\alpha}{\Delta(\alpha)} &\{(\alpha_{12} i h_-(\alpha) + \alpha_{22} g(\alpha))\Psi_{1\alpha} \\ &+ (i\alpha - i\alpha_{11} h_-(\alpha) - \alpha_{12} g(\alpha))\Psi_{2\alpha}\}, \\ \tau_\alpha = \frac{i\alpha}{\Delta(\alpha)} &\{(\alpha_{11} g(\alpha) + \alpha_{12} i h_+(\alpha))\Psi_{2\alpha} \\ &- (i\alpha + \alpha_{12} g(\alpha) + \alpha_{22} i h_+(\alpha))\Psi_{1\alpha}\}, \end{aligned}$$

where

$$\begin{aligned}
\Delta(\alpha) &= -[\alpha_{11}g(\alpha) + \alpha_{12}ih_+(\alpha)][\alpha_{12}ih_-(\alpha) + \alpha_{22}g(\alpha)] \\
&\quad -[i\alpha + \alpha_{12}g(\alpha) + \alpha_{22}ih_+(\alpha)][i\alpha - \alpha_{11}ih_-(\alpha) - \alpha_{12}g(\alpha)] \\
&= \alpha^2 + \{g^2(\alpha) + h_+(\alpha)h_-(\alpha)\}\{\alpha_{12}^2 - \alpha_{11}\alpha_{22}\} \\
&\quad +\alpha_{22}\alpha h_+(\alpha) - \alpha_{11}\alpha h_-(\alpha).
\end{aligned} \tag{5.6.16}$$

Applying the inverse Fourier transform, using (5.6.15) and changing the order of integration we get the integral representations of traction components

$$\begin{aligned}
\tau(x) &= \int_{-c}^{c} \{\Psi_1(\xi)l_{11}(\xi - x) + \Psi_2(\xi)l_{12}(\xi - x)\}\, d\xi, && (5.6.17)\\
\sigma(x) &= \int_{-c}^{c} \{\Psi_1(\xi)l_{21}(\xi - x) + \Psi_2(\xi)l_{22}(\xi - x)\}\, d\xi, \quad |x| < \infty,
\end{aligned}$$

where

$$l_{kj}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p_{kj}(\alpha)e^{i\alpha t}d\alpha, \tag{5.6.18}$$

$$\begin{aligned}
p_{11}(\alpha) &= \frac{\alpha^2}{\Delta(\alpha)}\left\{1 + \frac{\alpha_{12}}{i\alpha}g(\alpha) + \frac{\alpha_{22}}{\alpha}h_+(\alpha)\right\}, \\
p_{12}(\alpha) &= \frac{i\alpha}{\Delta(\alpha)}\left\{\alpha_{11}g(\alpha) + \alpha_{12}ih_+(\alpha)\right\}, && (5.6.19)\\
p_{21}(\alpha) &= \frac{-i\alpha}{\Delta(\alpha)}\left\{\alpha_{12}ih_-(\alpha) + \alpha_{22}g(\alpha)\right\}, \\
p_{22}(\alpha) &= \frac{\alpha^2}{\Delta(\alpha)}\left\{1 - \frac{\alpha_{11}}{\alpha}h_-(\alpha) - \frac{\alpha_{12}}{i\alpha}g(\alpha)\right\}.
\end{aligned}$$

Considering the system (5.6.17) on the interval $(-c, c)$ and using (5.6.1) we obtain the system of integral equations

$$\boxed{\sum_{j=1}^{2}\int_{-c}^{c}\Psi_j(\xi)l_{kj}(\xi - x)d\xi = f_k(x), \quad k = 1,\ 2, \quad |x| < c.} \tag{5.6.20}$$

5.6.4 Analysis of the system of integral equations

First, we analyse the asymptotic behaviour of the kernel functions $l_{jk}(\xi - x)$ as $\xi \to x$. We note that $l_{jk}(t)$ are singular at $t = 0$. To obtain the structure of the singular terms, we need the asymptotics of the functions $p_{km}(\alpha)$ (see (5.6.18)) as

$\alpha \to 0$ or $\alpha \to \infty$. Direct calculations show that

$$
\begin{aligned}
g(\alpha) &\sim \frac{6}{\alpha^2}\left(\frac{\kappa_2^+}{a^2} - \frac{\kappa_2^-}{b^2}\right),\\
h_+(\alpha) &\sim \frac{12}{\alpha^3}\left(\frac{\kappa_2^+}{a^3} + \frac{\kappa_2^-}{b^3}\right),\\
h_-(\alpha) &\sim -\frac{4}{\alpha}\left(\frac{\kappa_2^+}{a} + \frac{\kappa_2^-}{b}\right),
\end{aligned}
$$

as $\alpha \to 0$.

It follows from (5.6.16) and (5.6.19) that

$$\Delta(\alpha) = O(\alpha^{-4}), \quad \alpha \to 0$$

and

$$
\begin{aligned}
p_{11}(\alpha) = O(\alpha^2) \quad &, \quad p_{12}(\alpha) = O(\alpha^2),\\
p_{21}(\alpha) = O(\alpha^3) \quad &, \quad p_{22}(\alpha) = O(\alpha^3), \quad \alpha \to 0.
\end{aligned}
\tag{5.6.21}
$$

The relationships (5.6.21) provide the convergence of the integrals (5.6.18) in the vicinity of $\alpha = 0$, uniformly with respect to t. On the other hand, at infinity we have

$$g(\alpha) \sim \Lambda_-, \quad h_+(\alpha) \sim \pm\Lambda_+, \quad h_-(\alpha) \sim \mp\Lambda_+, \quad \alpha \to \pm\infty,$$

where

$$\Lambda_+ = 2(\kappa_2^+ + \kappa_2^-), \quad \Lambda_- = \kappa_1^+ - \kappa_1^-.$$

The function $\Delta(\alpha)$ is characterised by

$$
\begin{aligned}
\Delta(\alpha) \sim \alpha^2 + (\alpha_{11} + \alpha_{22})\Lambda_+|\alpha| + (\Lambda_+^2 - \Lambda_-^2) \ \ &(\alpha_{11}\alpha_{22} - \alpha_{12}^2),\\
&\alpha \to \pm\infty.
\end{aligned}
$$

Hence, the functions p_{kj} possess the asymptotics

$$
\begin{aligned}
p_{11}(\alpha) &= 1 + \frac{\alpha_{12}}{i\alpha}\Lambda_- - \frac{\alpha_{11}}{|\alpha|}\Lambda_+ + O(\frac{1}{\alpha^2}),\\
p_{22}(\alpha) &= 1 - \frac{\alpha_{12}}{i\alpha}\Lambda_- - \frac{\alpha_{22}}{|\alpha|}\Lambda_+ + O(\frac{1}{\alpha^2}),\\
p_{12}(\alpha) &= \frac{i\alpha_{11}}{\alpha}\Lambda_- - \frac{\alpha_{12}}{|\alpha|}\Lambda_+ + O(\frac{1}{\alpha^2}),\\
p_{21}(\alpha) &= -\frac{i\alpha_{22}}{\alpha}\Lambda_- - \frac{\alpha_{12}}{|\alpha|}\Lambda_+ + O(\frac{1}{\alpha^2}).
\end{aligned}
$$

Taking into account the relationship

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha t} d\alpha = \delta(t),
$$

where $\delta(t)$ is the Delta function, we obtain

$$
l_{kk}(t) = \delta(t) + l^0_{kk}(t), \quad k = 1,\ 2,
$$

where

$$
\begin{aligned}
l^0_{kk}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} p^0_{kk}(\alpha) e^{i\alpha t} d\alpha,\\
p^0_{kk}(\alpha) &= p_{kk}(\alpha) - 1.
\end{aligned}
$$

We note that $l^0_{11}(t)$, $l^0_{22}(t)$, $l_{12}(t)$, $l_{21}(t)$ have logarithmic singularities as $t \to 0$. To obtain the logarithmic terms explicitly, we write

$$
\begin{aligned}
l^0_{11}\ (t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} p^0_{11} e^{i\alpha t} d\alpha = \frac{1}{2\pi}\int_{-1}^{1} p^0_{11} e^{i\alpha t} d\alpha\\
&+ \ \frac{1}{2\pi}\int_L (p^0_{11} - \frac{\alpha_{12}}{i\alpha}\Lambda_- + \frac{\alpha_{11}}{|\alpha|}\Lambda_+) e^{i\alpha t} d\alpha\\
&+ \ \frac{1}{2\pi}\int_L (\frac{\alpha_{12}}{i\alpha}\Lambda_- - \frac{\alpha_{11}}{|\alpha|}\Lambda_+) e^{i\alpha t} d\alpha,
\end{aligned}
$$

where $L = (-\infty, -1) \bigcup (1, +\infty)$. The first two integrals in the right-hand side converge uniformly for all real t, whereas the last integral diverges at $t = 0$. Namely,

$$
\frac{1}{2\pi}\int_L \frac{e^{i\alpha t}}{i\alpha} d\alpha = \frac{1}{\pi}\text{sgn}\ t \int_{|t|}^{\infty} \frac{\sin\tau}{\tau} d\tau = -\frac{1}{\pi}\text{sgn}\ t\ \ \text{si}\ (|t|),
$$

$$\frac{1}{2\pi}\int_L \frac{e^{i\alpha t}}{|\alpha|}d\alpha = \frac{1}{\pi}\int_{|t|}^{\infty}\frac{\cos\tau}{\tau}d\tau = -\frac{1}{\pi}\mathrm{ci}\ (|t|),$$

where $\mathrm{si}(x)$, $\mathrm{ci}(x)$ are the sine and cosine integral functions given by

$$\mathrm{si}\ (x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k-1}}{(2k-1)(2k-1)!},$$
$$\mathrm{ci}\ (x) = \gamma + \log|x| + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2k(2k)!},$$

and γ is the Euler constant. Thus,

$$l_{11}(t) = \delta(t) + \frac{1}{\pi}\alpha_{11}\Lambda_+\log|t| + \tfrac{1}{2}\alpha_{12}\Lambda_-\mathrm{sgn}\ t + \tilde{l}_{11}(t),$$

where $\tilde{l}_{11}(t)$ is continuous on the whole real axis. In a similar way,

$$\begin{aligned} l_{22}(t) &= \delta(t) + \frac{1}{\pi}\alpha_{22}\Lambda_+\log|t| - \tfrac{1}{2}\alpha_{12}\Lambda_-\mathrm{sgn}\ t + \tilde{l}_{22}(t),\\ l_{12}(t) &= \frac{1}{\pi}\alpha_{12}\Lambda_+\log|t| - \tfrac{1}{2}\alpha_{11}\Lambda_-\mathrm{sgn}\ t + \tilde{l}_{12}(t),\\ l_{21}(t) &= \frac{1}{\pi}\alpha_{12}\Lambda_+\log|t| + \tfrac{1}{2}\alpha_{22}\Lambda_-\mathrm{sgn}\ t + \tilde{l}_{21}(t), \end{aligned}$$

where the functions $\tilde{l}_{kj}(t)$ are continuous. Hence, the system (5.6.20) can be written in the form

$$\begin{aligned} &\Psi_k(cx) + c\sum_{j=1}^{2}\int_{-1}^{1}\{\frac{\beta_{kj}}{\pi}\log|\xi - x| + \mathcal{K}_{kj}(\xi - x)\}\Psi_j(c\xi)d\xi\\ &= f_k(cx),\ \ |x|<1,\ \ k=1,\ 2, \end{aligned} \tag{5.6.22}$$

which is a Fredholm system of the second kind. Here

$$\begin{aligned}
\mathcal{K}_{kj}(t) &= \frac{\beta_{kj}}{\pi}\log c + \frac{1}{2\pi}\int_{-\infty}^{\infty} p^1_{kj}(\alpha)e^{i\alpha tc}d\alpha \\
&+ \frac{(-1)^k}{\pi}\eta_{kj}\ \operatorname{sgn} t \ \operatorname{si}\ (c|t|) \\
&+ \frac{\beta_{kj}}{\pi}(\gamma + \sum_{s=1}^{\infty}\frac{(-1)^s(ct)^{2s}}{2s(2s)!}), \\
p^1_{kj}(\alpha) &= p_{kj}(\alpha) - \delta_{kj} + \frac{(-1)^k\eta_{kj}}{i\alpha}\varepsilon(\alpha) + \frac{\beta_{kj}\varepsilon(\alpha)}{|\alpha|}, \\
\varepsilon &= \begin{cases} 1, & |\alpha| > 1 \\ 0, & |\alpha| < 1, \end{cases} \\
\beta_{kj} &= \alpha_{kj}\Lambda_+, \\
\eta_{11} &= \alpha_{12}\Lambda_-, \quad \eta_{12} = \alpha_{11}\Lambda_-, \\
\eta_{21} &= \alpha_{22}\Lambda_-, \quad \eta_{22} = \alpha_{12}\Lambda_-.
\end{aligned}$$

We note that $p^1_{kj}(\alpha) = O(\alpha^{-2})$ as $\alpha \to \pm\infty$.

5.6.5 Boundedness of the solution

To prove the boundedness of the solution at the point $x = 1$ we allow an integrable singularity for the functions $\Psi_k(cx)$ at this point

$$\Psi_k(cx) \sim \mathcal{N}_k(1-x)^\alpha, \quad x \to 1-0; \quad -1 < \alpha < 0. \tag{5.6.23}$$

The behaviour of the integral with the logarithmic kernel at the end $x = 1$ is described by

$$\int_{-1}^{1} \log|y-x|(1-y)^\alpha dy = \frac{\pi\cot\pi\alpha}{\alpha+1}(1-x)^{\alpha+1} + \phi(x;\alpha), \quad x \to 1-0, \tag{5.6.24}$$

where $\phi(x;\alpha)$ is bounded in the neighbourhood of the point $x = 1$. Then we substitute formulae (5.6.23) and (5.6.24) into (5.6.22) to get

$$\mathcal{N}_k(1-x)^\alpha + c\sum_{j=1}^{N}\{\beta_{kj}\frac{\cot\pi\alpha}{\alpha+1}\mathcal{N}_j(1-x)^{\alpha+1} + \phi_{kj}(x;\alpha)\} = f_k(cx), \quad x \to 1-0,$$

where $\phi_{kj}(x;\alpha)$ are bounded as $x \to 1$. The left hand-sides of the last equations are bounded if and only if $\mathcal{N}_1 = \mathcal{N}_2 = 0$. It means that the functions $\Psi_k(cx)$ may possess logarithmic singularities or be bounded at the end $x = 1$. Again, we allow

$$\Psi_k(cx) \sim M_k \ln(1-x), \quad x \to 1-0$$

and take into account the behaviour of the integral

$$\int_{-1}^{1} \log|y-x| \log(1-y) dy = C(1-x)\ln(1-x), +\phi_0(x), \quad x \to 1-0 \tag{5.6.25}$$

where C is a constant and $\phi_0(x)$ is a bounded function as $x \to 1$. In a similar way, system (5.6.22) and formula (5.6.25) give $M_1 = M_2 = 0$.

Thus, this analysis shows that the functions $\Psi_1(cx)$, $\Psi_2(cx)$ possess neither logarithmic nor power singularities at the end $x = 1$ and $x = -1$ (obviously, the analysis of the functions Ψ_1, Ψ_2 at the point $x = -1$ is similar to the previous one for $x = 1$), i.e. the solution of (5.6.22) is bounded at the ends of the interval $(-1, 1)$. Moreover, if we take into account the interface conditions (5.6.13), the traction conditions (5.6.1) and the system (5.6.22), we find that the displacement jumps $\chi_1(x) = [u](x)$ and $\chi_2(x) = [v](x)$ are continuous at the points $x = \pm c$. The stress components $\sigma_y(x,0)$ and $\tau_{xy}(x,0)$ are bounded and discontinuous at the ends $x = \pm c$.

5.7 Summary and numerical results

First, we summarise the results of the analysis of the scalar formulations (see Sections 5.3-5.5) associated with the heat transfer problems (or anti-plane shear). We have presented the exact solutions of problems on cracks along imperfect interface boundaries. The author is not aware of similar results published in the literature relevant to this work. The asymptotic analysis shows that the solution (the temperature or the transverse displacement) is bounded and its tangential derivative is characterised by a weak logarithmic singularity at the crack tip; the normal derivative of the solution is bounded. The behaviour of temperature (or displacement jump) at infinity depends on the geometry of the whole domain and the type of boundary conditions on the exterior contour. We have shown that the case of an infinite plane is qualitatively different from the cases involving a strip with a crack: in the latter case the solution is either bounded or decays exponentially at infinity, whereas for the problem involving an infinite two-phase plane we deal with power asymptotic expansions at infinity. In particular, the constant A in formula (5.4.8) has been evaluated explicitly to characterise the displacement (or temperature) jump at infinity along the crack when the Neumann boundary conditions are specified on the upper and lower boundaries of the strip. This problem can be considered as a model boundary layer formulation for a singularly perturbed domain involving a crack in a thin rectangle.

Next, we analyse a plane elasticity problem for a two-phase strip with a finite crack. A numerical solution is presented for the system (5.6.22). The numerical algorithm employed here is described in Section 5.7.1.

The elastic layers are characterised by the Young's moduli $E_\pm$ and by the values $\nu_\pm$ of the Poisson ratio. By $\lambda_\pm$, $\nu_\pm$, we denote the Lamé constants of the elastic materials given by

$$\lambda_\pm = \frac{E_\pm \nu_\pm}{(1+\nu_\pm)(1-2\nu_\pm)}, \quad \mu_\pm = \frac{E_\pm}{2(1+\nu_\pm)},$$

while

$$\lambda = \frac{E_0 \nu_0}{(1+\nu_0)(1-2\nu_0)}, \quad \mu = \frac{E_0}{2(1+\nu_0)}$$

are the normalised Lamé constants related to the middle layer. For all the tests we considered the upper material to be aluminium with the following elastic moduli (see Adams and Wake (1995) and Section 2.4.7)

$$E_{+(Al)} = 70 \;\; GPa; \quad \nu_{+(Al)} = 0.3 \, .$$

The lower material can be either aluminium, CFRP (Carbon Fibre Reinforced Laminates) or brass, the last two having the following elastic moduli

$$E_{-(CFRP)} = 135 \;\; GPa; \quad \nu_{-(CFRP)} = 0.3,$$

$$E_{-(Br)} = 100 \;\; GPa; \quad \nu_{-(Br)} = 0.25 \, .$$

The interface layer is assumed to be made of the adhesive FM 1000 and characterised by the normalised moduli,

$$E = 10 \;\; GPa; \quad \nu = 0.4$$

Case	*Type of Load*	*Lower layer*
A.1	$f_1 = 0;\ f_2 = -1$	CFRP
A.2	$f_1 = 0;\ f_2 = -1$	Brass
A.3	$f_1 = 1;\ f_2 = 0$	CFRP
A.4	$f_1 = 1;\ f_2 = 0$	Brass

Table 5.1: Symmetric case ($a = b = c = 1$).

whereas the real values are given by

$$E_0 = 1.24\ \ GPa; \quad \nu_0 = 0.41$$

and thus here $\epsilon = 0.124$. The parameters that are involved in the interface condition in this case have the following values,

$$\alpha_{11} = 0.35\ \ GPa; \quad \alpha_{22} = 2.3\ \ GPa$$

and by the approximation (5.2.9) (see Section 5.7.1), $\alpha_{12} = \alpha_{21} = 0$.

Several cases of applied load are considered and listed in the Tables 5.1 and 5.2.

Figures 5.7 and 5.9 include the graphs of the displacement jump components evaluated for the symmetric structure involving the elastic layers of the same thickness.

The results are presented for the normal and shear external loads and different values of elastic moduli. We note that the negative values of the vertical displacement jump correspond to overlapping of the phases on the imperfect interface. This is a consequence of the linearisation of the model. However, both negative and positive values of the tangential displacement jump presented in Figures 5.7 and 5.9 make sense physically. It is noted that the displacement jump is continuous in the vicinity of the crack ends.

The corresponding graphs for stress are given in Figures 5.8 and 5.10. As predicted the stress components $\sigma_y(x, 0), \tau_{xy}(x, 0)$ are bounded and discontinuous at the ends of the crack.

Figures 5.11 and 5.12 show the graphs of displacement and traction components for the case of layers of different thickness and elastic moduli. Only shear load cases are considered. It is observed that the longitudinal displacement jump takes its maximum value for the case of a symmetric strip when both layers have the same thickness and elastic moduli.

Case	*Geometry*	*Lower layer*
B.1	$a = c = 1;\ b = 100$	Aluminium
B.2	$a = b = c = 1$	Aluminium
B.3	$a = c = 1;\ b = 100$	Brass
B.4	$a = b = c = 1$	Brass

Table 5.2: Combined Effect ($f_1 = 1;\ f_2 = 0$).

The examples presented above are given for the purpose of illustration yet the algorithm has been designed for a general smooth load and can take into account a wide variety of geometric parameters of the structure.

5.7.1 Numerical approach

Here we present a brief description of the numerical approach that we use to solve the system (5.6.22) that is a Fredholm system of integral equations of the second kind. Writing

$$\begin{aligned} \Psi(cx_n) &= A_n^{(k)}, \quad k = 1, 2, \quad n = 1, 2, \dots, N, \\ x_n &= -1 + \frac{2n-1}{N}, \quad n = 1, 2, \dots, N, \\ y_n &= -1 + \frac{2n}{N}, \quad n = 0, 1, \dots, N, \end{aligned}$$

one can discretise the system (5.6.22) as follows:

$$\Psi_k(cx_n) + c\sum_{j=1}^{2}\sum_{m=1}^{N}\int_{y_{m-1}}^{y_m} \{\frac{\beta_{kj}}{\pi}\ln|\xi - x_n| + \mathcal{K}_{kj}(\xi - x_n)\}d\xi\Psi_j(cx_m) = f_k(cx_n).$$

Thus, we arrive at the following linear system of algebraic equations

$$A_n^{(k)} + c\sum_{j=1}^{2}\sum_{m=1}^{N}\mathcal{D}_{nm}^{(k,j)}A_m^{(j)} = c_n^{(k)}, \quad k = 1, 2; \quad n = 1, 2, \dots, N \tag{5.7.1}$$

where the coefficients $\mathcal{D}_{nm}^{(k,j)}$ and $c_n^{(k)}$ are given by

$$\begin{aligned} \mathcal{D}_{nm}^{(k,j)} &= \int_{y_{m-1}}^{y_m} [\frac{\beta_{kj}}{\pi}\ln|\xi - x_n| + \mathcal{K}_{kj}(\xi - x_n)]d\xi \qquad (5.7.2) \\ c_n^{(k)} &= f_k(cx_n), \quad k = 1,\ 2. \end{aligned}$$

Direct calculations show that (5.7.2) can be written in the following way,

$$\mathcal{D}_{nm}^{(k,j)} = \frac{\beta_{kj}}{\pi}[(x_n - y_{m-1})(\ln|x_n - y_{m-1}| - 1) - (x_n - y_m)(\ln|x_n - y_m| - 1)]$$
$$+ \frac{2\log c}{\pi N}\beta_{kj} + \frac{1}{\pi}\int_{-\infty}^{\infty} Q(\alpha)d\alpha + \frac{(-1)^k\eta_{kj}}{\pi}[\frac{\pi}{N}W_{mn} + S_1] + \frac{\beta_{kj}}{\pi}[\frac{2\gamma}{N} + S_2],$$

where

$$S_1 = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}c^{2s-1}[(y_m - x_n)^{2s} - (y_{m-1} - x_n)^{2s}]}{(2s-1)(2s)!},$$
$$S_2 = \sum_{s=1}^{\infty} \frac{(-1)^{s}c^{2s}[(y_m - x_n)^{2s+1} - (y_{m-1} - x_n)^{2s+1}]}{2s(2s+1)!},$$
$$Q(\alpha) = \left\{p_{kj}(\alpha) - \delta_{kj} + (-1)^k\eta_{kj}\frac{\varepsilon(\alpha)}{i\alpha} + \beta_{kj}\frac{\varepsilon(\alpha)}{|\alpha|}\right\}\frac{\sin(\frac{\alpha c}{N})}{\alpha}e^{i\alpha c(t_m - x_n)},$$
$$\delta_{kj} = \begin{cases} 1 & , k = j \\ 0 & , k \neq j \end{cases}, W_{mn} = \begin{cases} 1 & , x_n > y_m \\ -1 & , x_n < y_{m-1} \\ 0 & , x_n \in (y_{m-1}, y_m) \end{cases},$$
$$t_m = -1 + \frac{2m-1}{N}.$$

The value of $\int_{-\infty}^{\infty} Q(\alpha)d\alpha$ can be obtained by using any numerical procedure. Here we use the trapezoidal method. It is worth mentioning that $Q(\alpha) \in L_1(-\infty, +\infty)$ and $Q(\alpha) = O(\frac{1}{\alpha^3})$ as $\alpha \to \pm\infty$. The structure of the functions $p_{kj}(\alpha), \Delta(\alpha), g(\alpha)$ and $h_\pm(\alpha)$ (see (5.6.19), (5.6.16), (5.6.11) and (5.6.12)) allows us to write them in the form,

$$p_{kj}(\alpha) = p_{kj}^e(\alpha) + ip_{kj}^o(\alpha), \tag{5.7.3}$$

where the real functions p_{kj}^e are even and the real functions p_{kj}^o are odd,

$$\begin{array}{ll}
p_{11}^e(\alpha) = \frac{\alpha^2}{\Delta(\alpha)}[\frac{\alpha_{22}}{\alpha}h_+(\alpha) + 1]\ ; & p_{11}^o(\alpha) = -\frac{\alpha}{\Delta(\alpha)}\alpha_{12}g(\alpha) \\
p_{12}^e(\alpha) = -\frac{\alpha}{\Delta(\alpha)}\alpha_{12}h_+(\alpha)\ ; & p_{12}^o(\alpha) = \frac{\alpha}{\Delta(\alpha)}\alpha_{11}g(\alpha) \\
p_{21}^e(\alpha) = \frac{\alpha}{\Delta(\alpha)}\alpha_{12}h_-(\alpha)\ ; & p_{21}^o(\alpha) = -\frac{\alpha}{\Delta(\alpha)}\alpha_{22}g(\alpha) \\
p_{22}^e(\alpha) = \frac{\alpha^2}{\Delta(\alpha)}[1 - \frac{\alpha_{11}}{\alpha}h_-(\alpha)]\ ; & p_{22}^o(\alpha) = \frac{\alpha}{\Delta(\alpha)}\alpha_{12}g(\alpha).
\end{array}$$

Thus, it turns out can that all the values $\mathcal{D}_{nm}^{(k,j)}$ in the system (5.7.1) are real.

We substitute (5.7.3) into the relationship for $Q(\alpha)$ and obtain

$$\int_{-\infty}^{\infty} Q(\alpha)d\alpha = 2\int_{0}^{\infty} Q_{+}(\alpha)d\alpha,$$

where

$$Q_{+}(\alpha) = \left\{[p_{kj}^{e}(\alpha) - \delta_{kj} + \beta_{kj}\frac{\varepsilon(\alpha)}{\alpha}]\cos c(x_m - x_n)\alpha \right.$$
$$\left. +[-p_{kj}^{o}(\alpha) + (-1)^k \eta_{kj}\frac{\varepsilon(\alpha)}{\alpha}]\sin c(x_m - x_n)\alpha \right\} \frac{\sin(\frac{\alpha c}{N})}{\alpha}.$$

5.7.2 The displacement jump on the whole imperfect interface

Once the system (5.7.1) is solved we can then approximate the values for the displacement jumps $[u](cx_n)$ and $[v](cx_n)$ along the crack,

$$\boxed{[u](cx_n) = \tfrac{1}{\alpha_*}\left\{\alpha_{22}[c_n^{(1)} - A_n^{(1)}] - \alpha_{12}[c_n^{(2)} - A_n^{(2)}]\right\},}$$

$$\boxed{[v](cx_n) = \tfrac{1}{\alpha_*}\left\{-\alpha_{12}[c_n^{(1)} - A_n^{(1)}] + \alpha_{11}[c_n^{(2)} - A_n^{(2)}]\right\},}$$

where α_{11}, α_{12}, α_{21} and α_{22} are the same as in Section 5.6 (see (5.6.2)) and $\alpha_* = \alpha_{11}\alpha_{22} - \alpha_{12}^2$.

For $|x| > c$, the components of traction vector along the interface are

$$\tau_{xy} = f_1(x), \quad \sigma_y = f_2(x),$$

where

$$f_k(\pm cz_n) = c\sum_{j=1}^{2}\sum_{m=1}^{N} A_m^{(j)}\mathcal{D}_{nm}^{(k,j)\pm} \tag{5.7.4}$$

and $\mathcal{D}_{nm}^{(k,j)\pm}$ coincides with $\mathcal{D}_{nm}^{(k,j)}$ when one substitutes x_n by $\pm z_n$, $z_n > 1$.

Finally, we can also find the displacement jumps outside the crack,

$$\boxed{[u](\pm cz_n) = \tfrac{1}{\alpha_*}\left\{\alpha_{22}f_1(\pm cz_n) - \alpha_{12}f_2(\pm cz_n)\right\},}$$

$$\boxed{[v](\pm cz_n) = \frac{1}{\alpha_*}\left\{-\alpha_{12} f_1(\pm cz_n) + \alpha_{11} f_2(\pm cz_n)\right\}.}$$

For all the calculations, the matrix α_{ij} is approximated by the matrix obtained in (5.2.9) by which it can be seen that $\alpha_{12} = \alpha_{21}$ and

$$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}.$$

The quantities μ and λ are the normalised values of the elastic moduli for the adhesive joint treated in Section 5.2.

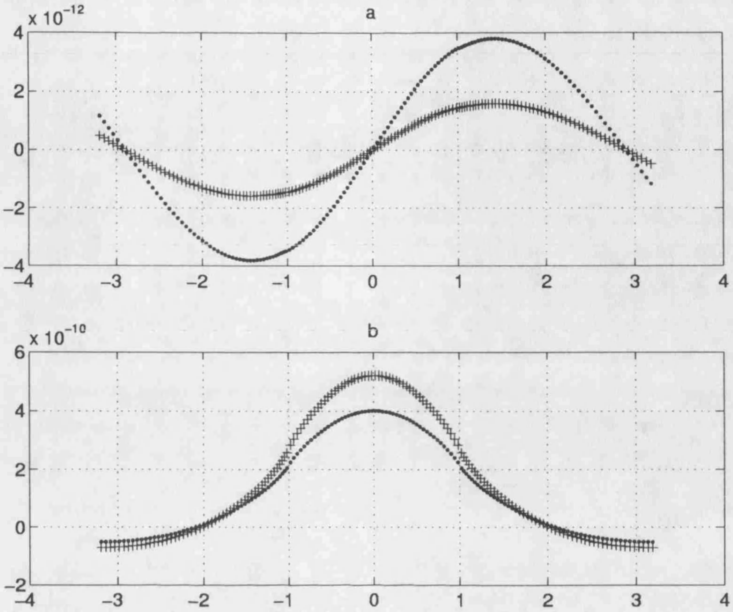

Figure 5.7: The displacement jump vrs. x for the cases (see Table 5.1):
a: ... $[u]$(A.1); +++ $[u]$(A.2);
b: ... $[v]$(A.1); +++ $[v]$(A.2).

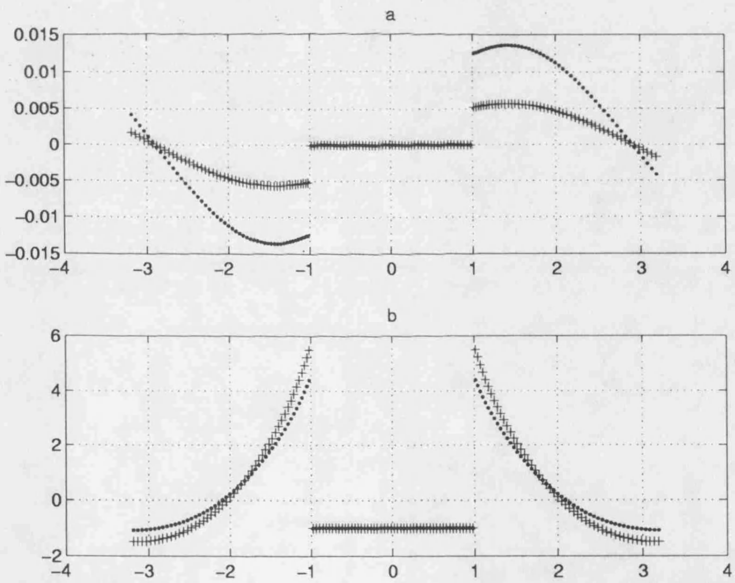

Figure 5.8: The stress components along the interface vrs. x for the cases :
a: ... τ_{xy}(A.1); +++ τ_{xy}(A.2);
b: ... σ_y(A.1); +++ σ_y(A.2).

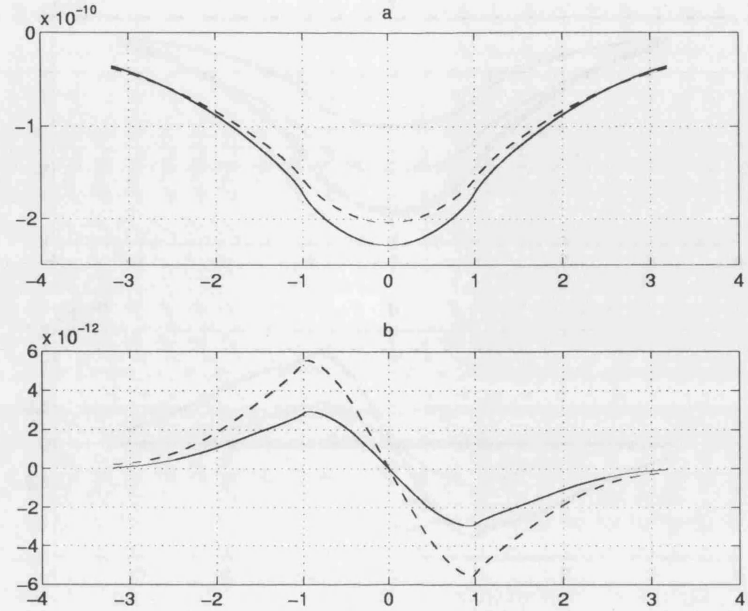

Figure 5.9: The displacement jump vrs. x for the cases (see Table 5.1):
a: - -$[u]$(A.3); —$[u]$ (A.4);
b: - -$[v]$(A.3); —$[v]$ (A.4).

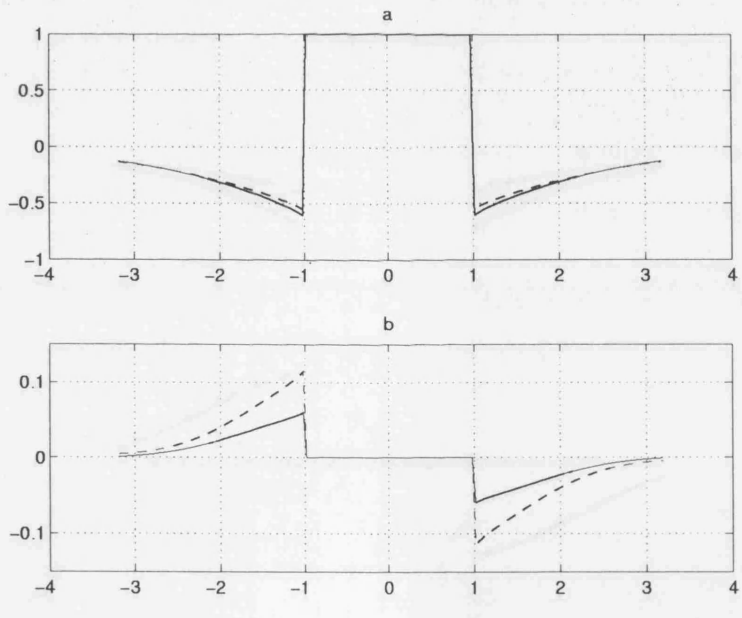

Figure 5.10: The stress components along the interface vrs. x for the cases :
a: - -τ_{xy}(A.3); —τ_{xy} (A.4);
b: - -σ_y(A.3); —σ_y (A.4).

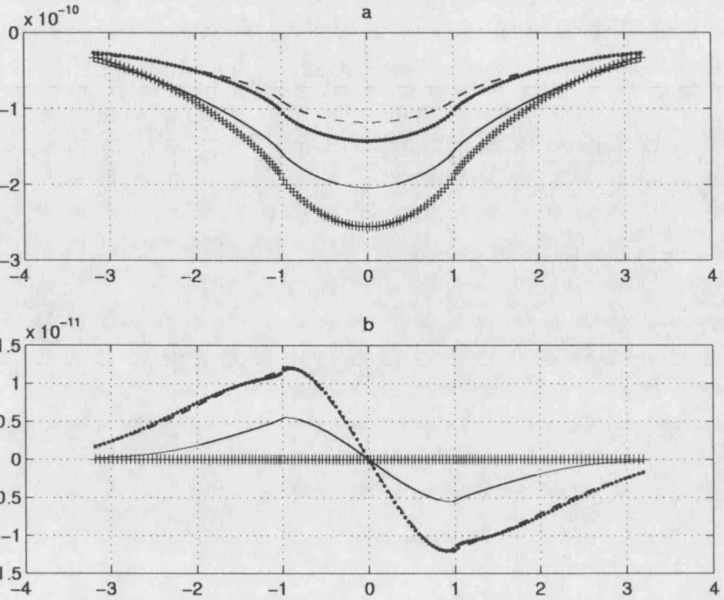

Figure 5.11: The displacement jump vrs. x for the cases (see Table 5.2):
a: ... $[u]$(B.1); +++ $[u]$(B.2); - -$[u]$(B.3); —$[u]$ (B.4);
b: ... $[v]$(B.1); +++ $[v]$(B.2); - -$[v]$(B.3); —$[v]$ (B.4).

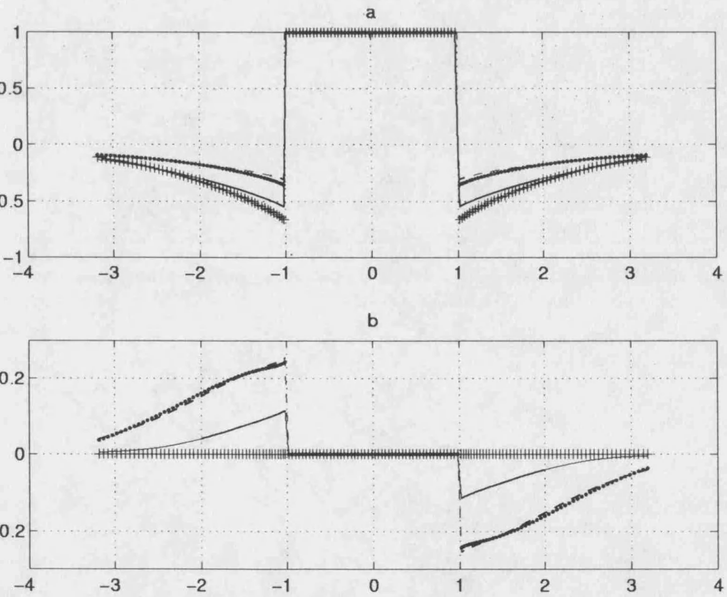

Figure 5.12: The stress components along the interface vrs. x for the cases :
a: ... τ_{xy}(B.1); +++ τ_{xy}(B.2); - -τ_{xy}(B.3); —τ_{xy} (B.4);
b: ... σ_y(B.1); +++ σ_y(B.2); - -σ_y(B.3); —σ_y (B.4).

Bibliography

Achenbach, J. D. (1973). *Wave propagation in elastic solids.* North Holland Publishing, Amsterdam, London.

Adams, R. D. and Wake, W. C. (1995). *Structural Adhesive Joints in Engineering.* Elsevier, New York, London.

Ahlfors, L. V. (1966). *Complex analysis.* McGraw-Hill Book Co, New York.

Antipov, Y. A. (1993). An efficient solution of prandtl-type integrodifferential equations in a section and its application to contact problems for a strip. *Journal of Applied Mathematics and Mechanics*, vol.57, no.3, pp.547–556.

Antipov, Y. A., Avila-Pozos, O., Kolaczkowski, S. T., and Movchan, A. B. (1999a). Mathematical models of delamination cracks on imperfect interfaces. Submitted to Int. Journal of Solids and Structures.

Antipov, Y. A., Kolaczkowski, S. T., Movchan, A. B., and Spence, A. (1999b). Asymptotic analysis for cracks in a catalytic monolith combustor. Accepted in Int. Journal of Solids and Structures.

Arutyunyan, N. K., Movchan, A. B., and Nazarov, S. A. (1987). Behaviour of solutions of elasticity problems in noncompact domains with parabolic and cylindrical inclusions or cavities. *Advances in Mechanics*, vol.10, no.4, pp.3–91.

Avila-Pozos, O., Klarbring, A., and Movchan, A. B. (1999). Asymptotic model of orthotropic highly inhomogeneous layered structure. *Mechanics of Materials*, vol.31, pp.101–115.

Avila-Pozos, O. and Movchan, A. B. (1999). Asymptotic model for an oscillating inhomogeneous layered structure with an imperfect interface. To be submitted to Int. Journal of Solids and Structures.

Bakhvalov, N. S. and Panasenko, G. P. (1989). *Homogenization: Averaging processes in periodic media.* Kluwer, Dordrecht.

Basi, S., Rogers, T. G., and Spencer, A. J. M. (1991). Hygrothermoelastic analysis of anisotropic inhomogeneous and laminated plates. *Journal of the Mechanics and Physics of Solids*, vol.39, pp.1–22.

Baxter, S. C. and Horgan, C. O. (1995). End effects for anti-plane shear deformations of sandwich structures. *Journal of Elasticity*, vol.40, pp.123–164.

Baxter, S. C. and Horgan, C. O. (1997). Anti-plane shear deformations of anisotropic sandwich structures: end effects. *Int. Journal of Solids and Structures*, vol.34, no.1, pp.79–88.

Bensoussan, A., Lions, J. L., and Papanicolaou, G. (1978). *Asymptotic analysis for periodic structures.* North Holland publishing company, Amsterdam, New York, Oxford.

Benveniste, Y. (1984). On the effect of debonding on the overall behavior of composite materials. *Mechanics of Materials*, vol.3, pp.349–358.

Benveniste, Y. (1999). On the decay of end effects in conduction phenomena: A sandwich strip with imperfect interfaces of low or high conductivity. *Journal of Applied Physics*, vol.86, no.3, pp.1273–1279.

Bigoni, D., Ortiz, M., and Needleman, A. (1997). Effect of interfacial compliance on bifurcation of a layer bonded to a substrate. *Int. Journal of Solids and Structures*, vol.34, no.33–34, pp.4305–4326.

Bishop, J. A. (1997). The history of redux and the redux bonding process. *Int. Journal of Adhesion and Adhesives*, vol.17, pp.287–301.

Brekhovskikh, L. M. (1980). *Waves in Layered Media.* Academic Press, Inc, Orlando, USA.

Caillerie, D. (1980). The effect of a thin inclusion of high rigidity in an elastic body. *Mathematical Methods in the Applied Sciences*, vol.2, pp.251–270.

Caillerie, D. (1984). Thin elastic and periodic plates. *Mathematical Methods in the Applied Sciences*, vol.6, pp.159–191.

Calcote, L. R. (1969). *The analysis of laminated composite structures.* Van Nostrand Reinhold, New York.

Choi, I. and Horgan, C. O. (1977). Saint-Venant's principle and end effects in anisotropic elasticity. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.44, pp.424–430.

Choi, I. and Horgan, C. O. (1978). Saint-Venant end effects for plane deformations of sandwichs strips. *Int. Journal of Solids and Structures*, vol.14, pp.187–195.

Christensen, R. M. (1979). *Mechanics of composite mechanics.* Wiley, New York.

Ciarlet, P. G. (1990). *Plates and Junctions in Elastic Multi-Structures.* Masson, Paris.

Ciarlet, P. G. (1997). *Mathematical elasticity: Volume II: Theory of plates.* Elsevier, Amsterdam, Lausanne, New York, Oxford, Shannon, Tokyo.

Ciarlet, P. G. and Destuynder, P. (1979). A justification of the two-dimensional linear plate model. *Journal of Mécanique*, vol.18, pp.315–344.

Conway, J. B. (1978). *Functions of one complex variable.* Springer-Verlag, New York, Heidelberg, Berlin, 2nd edition.

Crafter, E. C., Heise, R. M., Horgan, C. O., and Simmonds, J. G. (1993). The eigenvalues for a self-equilibrated, semiinfinite, anisotropic elastic strip. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.60, no.2, pp.276–281.

Drumheller, D. S. and Bedford, A. (1973). On a continuum theory for a laminated medium. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.40, pp.527–532.

England, A. H. (1965). A crack between dissimilar media. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.E 32, pp.400–402.

Erdogan, F. (1963). Stress distribution in a non-homogeneous elastic plane with cracks. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.30, pp.232–236.

Erdogan, F. (1965). Stress distribution in bonded dissimilar materials with cracks. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.E 32, pp.418–423.

Gakhov, F. D. (1966). *Boundary Value Problems.* Pergamon Press, Oxford, London, Edinburgh, New York, Paris, Frankfurt, 2nd edition.

Gradshteyn, I. S. and Ryzhik, I. M. (1980). *Table of integrals, series and products.* Academic Press, New York, 4th edition.

Graff, K. F. (1975). *Wave motion in elastic solids.* Oxford University Press, London.

Hashin, Z. (1967). Plane anisotropic beams. *Journal of Applied Mechanics: Trans. ASME Series E*, vol., pp.257–262.

Hayes, R. E. and Kolaczkowski, S. T. (1997). *Introduction to catalytic combustion.* Gordon and Breach Publishers, New York.

Hegemier, G. A., Gurtman, G. A., and Nayfeh, A. H. (1973). A continuum theory for wave guide-type propagation in laminated and fiber reinforced composites. *Int. Journal of Solids and Structures*, vol.9, pp.395–414.

Horgan, C. O. (1989). Recent developments concerning Saint-Venant's principle: an update. *Appl. Mech. Rev*, vol.42, pp.295–303.

Horgan, C. O. (1995). Anti-plane shear deformations in linear and non-linear solid mechanics. *SIAM Rev.*, vol.37, pp.53–81.

Horgan, C. O. and Knowles, J. K. (1983). Recent developments concerning Saint-Venant's principle. In Hutchinson, J. W. and Wu, T. Y. (eds.), *Advances in Applied Mechanics*, pages 179–269. Academic Press, New York.

Horgan, C. O. and Miller, K. L. (1994). Anti-plane shear deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.61, pp.23–29.

Horgan, C. O. and Payne, L. E. (1993). On the asymptotic behavior of solutions of linear second-order boundary-value problems on a semi-infinite strip. *Archive for Rational Mechanics and Anal.*, vol.124, pp.277–303.

Horgan, C. O. and Simmonds, J. G. (1994). Saint-Venant's end effects in composite structures. *Comp. Engng.*, vol.3, pp.279–286.

Horvay, G., Ansia, A., and Esperanza, E. (1973). Transient heat conduction in laminated composites. *Journal of Heat Transfer: Trans. ASME Series C*, vol.95, pp.309–316.

Jikov, V. V., Kozlov, S. M., and Oleinik, O. A. (1995). *Homogenization of Differential Operators and Integral Functionals.* Springer, Berlin.

Jones, R. M. (1975). *Mechanics of composite materials.* Hemisphere, New York, Washington, Philadelphia, London.

Kalamkarov, A. L. and Kolpakov, A. G. (1997). *Analysis, design and optimization of composites structures.* Wiley, Chichester, New York, Toronto.

Kaprielian, P. V., Rogers, T. G., and Spencer, A. J. M. (1998). Theory of laminated elastic plates. i. isotropic laminae. *Phil Trans. Royal Society of London*, vol.A 324, pp.565–594.

Klarbring, A. (1991). Derivation of a model of adhesively bonded joints by the asymptotic expansion method. *Int. Journal of Engineering Sciences*, vol.29, no.4, pp.493–512.

Klarbring, A. and Movchan, A. B. (1995). Asymptotic analysis of adhesively bonded joints. Technical Report LiTH-IKP-R-848, Linköping University, Linköping, Sweden.

Klarbring, A. and Movchan, A. B. (1998). Asymptotic modelling of adhesive joints. *Mechanics of Materials*, vol.28, pp.137–145.

Kondrat'ev, V. A. (1967). Boundary problems for elliptic equations on domains with conical or angular points. *Transactions of the Moscow Mathematical Society*, vol.16, pp.227–313.

Kozlov, A., Maz'ya, V. G., and Movchan, A. B. (1999). Oxford mathematical monographs. In Ball, J. M., Friedlander, E. M., Macdonald, I. G., Nirenberg, L., Penrose, R., and Stewart, J. T. (eds.), *Asymptotic analysis of fields in multi-structures.* Oxford University Press, Oxford, New York.

Lekhnitskii, S. G. (1963). *Theory of elasticity in an anisotropic elastic body.* Holden-Day, San Francisco.

Lekhnitskii, S. G. (1968). *Anisotropic Plates.* Gordon and Breach Science Publishers, New York, London, Paris.

Leora, S. N., Nazarov, S. A., and Proskura, A. V. (1986). Derivation of limiting equations for elliptic problems in thin domains using computers. *Comput. Maths. Math. Phys.*, vol.26, no.4, pp.47–58.

Love, A. E. H. (1927). *The mathematical theory of elasticity.* Cambridge University Press, Cambridge, 4th edition.

Maheri, M. R. and Adams, R. D. (1998). On the flexural vibration of timoshenko beams and the applicability of the analysis to a sandwich configuration. *Journal of Sound and Vibration*, vol.209, no.3, pp.419–442.

Malyshev, B. M. and Salganik, R. L. (1965). The strength of adhesive joints using the theory of cracks. *Int. Journal of Fracture Mechanics*, vol.1, pp.114–128.

Maz'ya, V. G., Nazarov, S. A., and Plamenevsky, B. A. (1991). *Asymptotische Theorie Elliptischer Randwertanfgaben in Singülar Gestörten.* Akademie-Verlag, Berlin.

Miara, B. and Sanchez-Palencia, E. (1996). Asymptotic analysis of linearly elastic shells. *Asymptotic analysis*, vol.12, pp.41–54.

Miklowitz, J. (1980). *Elastic Waves and waveguides.* North-Holland Publishing, Amsterdam, New York, Oxford, 2nd edition.

Miller, K. L. and Horgan, C. O. (1995). Saint-Venant end effects for plane deformations of elastic composites. *Mech. Comp. Mat. Structures*, vol.2, pp.203–214.

Milton, G. W. (1986). Modelling the properties of composites by laminates. In Ericksen, J. L., Kinderlehrer, D., Kohn, R. V., and Lions, J. L. (eds.), *Homogenization and effective moduli of materials and media*, pages 150–174. Springer-Verlag, New York.

Movchan, A. B. and Movchan, N. V. (1995). *Mathematical Modelling of Solids with Non-regular Boundaries.* CRC, New York, London, Tokyo.

Movchan, A. B. and Nazarov, S. A. (1988). Stress-strain state of a plane with a thin elastic inclusion. *Mechanics of Solids*, vol.22, no.1, pp.73–83.

Muskhelishvili, N. I. (1953). *Some basic problems of the mathematical theory of elasticity.* Nordhoff, Groningen.

Nayfeh, A. H. (1974). Time-harmonic waves propagating normal to the layers of multilayered periodic media. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.41, pp.92–96.

Nayfeh, A. H. (1975). A continuum mixture theory of heat conduction in laminated composites. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.42, no.2, pp.399–404.

Nayfeh, A. H. (1980a). Continuum modelling of low frequency heat conduction in laminated composites with bonds. *Journal of Heat Transfer: Trans. ASME Series C*, vol.102, pp.312–319.

Nayfeh, A. H. (1980b). Simulation of the influence of bonding materials on electromagnetic wave propagation in laminated composites. *Journal of Applied Physics*, vol.51, no.6, pp.2987–2994.

Nayfeh, A. H. and Loh, J. J. (1977). On continuum mixture theory theories for shear wave propagation in trilaminated wave guides. *Journal of Sound and Vibration*, vol.55, pp.19–26.

Nayfeh, A. H. and Loh, J. J. (1979). Continuum modelling of electromagnetic waves in composite wave guides. *Journal of Applied Physics*, vol.50, no.2, pp.606–609.

Nayfeh, A. H. and Nassar, E. A. (1978). Simulation of the influence of bonding materials on the dynamic behavior of laminated wave guides. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.45, pp.822–828.

Nayfeh, A. H. and Nassar, E. A. (1982). The influence of bonding agents on the thermomechanically induced interfacial stresses in composite materials. *Fiber Science and Technology*, vol.16, pp.157–174.

Nayfeh, A. H. and Nassar, E. A. (1983). Horizontally polarized waves in composites with bonds. *Journal of Sound and Vibration*, vol.87, no.3, pp.439–447.

Nazarov, S. A. (1983a). Introduction to asymptotic methods of the theory of elasticity. Lecture notes, Leningr. University, Leningrad, USSR.

Nazarov, S. A. (1983b). The structure of solutions of elliptic boundary value problems in slender domains. *Vestnik Leningr. University*, vol.Ser. I, no.15, pp.99–104.

Nazarov, S. A. (1991). On three-dimensional effects near the vertex of a crack in a thin plate. *Journal of Applied Mathematics and Mechanics*, vol.55, no.3, pp.407–415.

Nazarov, S. A. and Plamenevsky, B. A. (1994). *Elliptic problems in domains with piecewise smooth boundaries.* Walter de Gruyter, Berlin, New York.

Nemat-Nasser, S. and Hori, M. (1992). *Micromechanics: Overall Properties of Heterogeneous Solids.* Dover.

Nguetseng, N. and Sanchez-Palencia, E. (1985). Stress concentration for defects distributed near a surface. In Ladevése, P. (ed.), *Local effects in the analysis of structures*, pages 55–74. Elsevier, Amsterdam, Netherlands.

Noble, B. (1958). *Methods based on the Wiener-Hopf technique: for the solution of partial differential equations.* Chelsea Publishing Company, New York, London, Paris, Los Angeles.

Ockendon, J. R., Howison, S. D., Lacey, A. A., and Movchan, A. B. (1999). *Applied partial differential equations.* Oxford University Press, Oxford.

Pagano, N. J. (1970). Exact solutions for rectangular bi-directional composites and sandwich plates. *J. Comp. Mats.*, vol.4, pp.20–35.

Postma, G. W. (1955). Wave propagation in a stratified medium. *Geophysics*, vol.20, pp.780–806.

Renardy, M. and Rogers, R. C. (1993). Texts in applied mathematics 13. In John, F., Marsden, J. E., Sirovich, L., Golubitski, M., and Jäger, W. (eds.), *An introduction to partial differential equations.* Springer-Verlag, New York.

Rice, J. R. and Sih, G. C. (1965). Plane problems of cracks in dissimilar media. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.E 32, pp.418–423.

Rogers, T. G. and Spencer, A. J. M. (1989). Thermoelastic stress analysis of moderately thick inhomogeneous and laminated plates. *IJSS*, vol.25, no.12, pp.1467–1482.

Rogers, T. G., Watson, P., and Spencer, A. J. M. (1992). An exact three-dimensional solution for normal loading of inhomogeneous and laminated anisotropic elastic plates of moderate thickness. *PTRSL*, vol.A 437, pp.199–213.

Rogers, T. G., Watson, P., and Spencer, A. J. M. (1995). Exact three-dimensional elasticity solutions for bending of moderately thick inhomogeneous and laminated strips under normal pressure. *Int. Journal of Solids and Structures*, vol.32, no.12, pp.1659–1673.

Sendeckyj, G. P. (1974). *Mechanics of Composite Mechanics.* Academic Press, New York, London.

Sih, G. C. (1978). *Mechanics of fracture.* Nordhoff International Publishing, Leyden.

Sih, G. C. and Chen, E. P. (1981). Cracks in composite materials. In Sih, G. C. (ed.), *Mechanics of fracture VI.* Martinus Nijhoff Publishers, Hague.

Sokolnikoff, I. S. (1956). *Mathematical Theory of Elasticity.* McGraw-Hill, New York.

Sorokin, S. V. (August 1995). Introduction to structural acoustics. Special report no. 28, Institute of Mechanical Engineering, Aalborg University.

Spencer, A. J. M. (1990). Three-dimensional elasticity solutions for stretching of inhomogeneous and laminated plates. *Mathematical Methods and Applications*, vol.32, no.12, pp.1659–1673.

Spencer, A. J. M., Watson, P., and Rogers, T. G. (1993). Exact theory of heterogeneous anisotropic elastic plates. In *Materials Science Forum Vol. 123–125, Proc. 7th Int. Symp. on Continuum Models of Discrete Systems*, pages 235–244.

Sun, C. T., Achenbach, J. D., and Herrmann, J. (1968). Continuum theory for a laminated medium. *Journal of Applied Mechanics: Trans. ASME Series E*, vol.90, pp.467–475.

Suquet, P. M. (1988). Discontinuities and plasticity. In Moreau, J. J. and Panagiotoupios, P. D. (eds.), *Nonsmooth Mechanics and Applications*, pages 279–340. Springer, New York, USA.

Tullini, N., Savoia, M., and Horgan, C. O. (1998). End effects for anti-plane shear deformations of periodically laminated strips with imperfect bonding. *Journal of Elasticity*, vol.50, pp.227–244.

Wijeyewickrema, A. C., Horgan, C. O., and Dundurs, J. (1996). Further analysis of end effects for plane deformations of sandwich strips. *Int. Journal of Solids and Structures*, vol.33, pp.4327–4336.

Willis, J. R. (1971). Fracture mechanics of interfacial cracks. *Journal of the Mechanics and Physics of Solids*, vol.19, pp.353–368.

Willis, J. R. (1972). The penny-shaped crack on an interface. *Quart. J. Mech. Appl. Math.*, vol.25, pp.367–385.

Wooley, G. R. and Carver, D. R. (1971). Stress concentration factors for bonded lap joints. *Journal of Aircraft*, vol.8, pp.817–820.

Zorin, I. S. and Nazarov, S. A. (1989). Edge effect in the bending of a thin three-dimensional plate. *Prikl. Matematics and Mechanics*, vol.53, no.4, pp.500–507.

Zorin, I. S. and Romashev, Y. A. (1988). On the state of stress and strain of layered plates of non-symmetric construction. *Prikl. Matematics and Mechanics*, vol.52, no.1, pp.67–74.